*Mathematical Physics for Quantum Science*
*Zhejiang University, 5 November 2024*

# Spectral continuum in the Rabi-Stark model

with Daniel Braak, Lei Cong, Hans-Peter Eckle, Elinor Twyeffort

D. Braak, L. Cong, H.-P. Eckle, HJ, E. K. Twyeffort
J. Opt. Soc. Am. B **41**, C97 (2024)
Feature issue: *"The Jaynes-Cummings model – 60 years and still counting"*
*arXiv:2403.16758*

Universität Augsburg University
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UNIVERSITY OF GOTHENBURG
University of Southampton

# Quantum Rabi model

$$H_R = \omega a^\dagger a + \Delta\sigma_z + g\sigma_x(a^\dagger + a)$$

photon frequency

level splitting
(two-level atom)

Rabi coupling

Isidor Rabi

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E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963)

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## "Rotating wave approximation"

neglecting rapidly oscillating terms in the interaction picture of $H_R$

OK if $g/\omega \ll 1$

## Jaynes-Cummings model

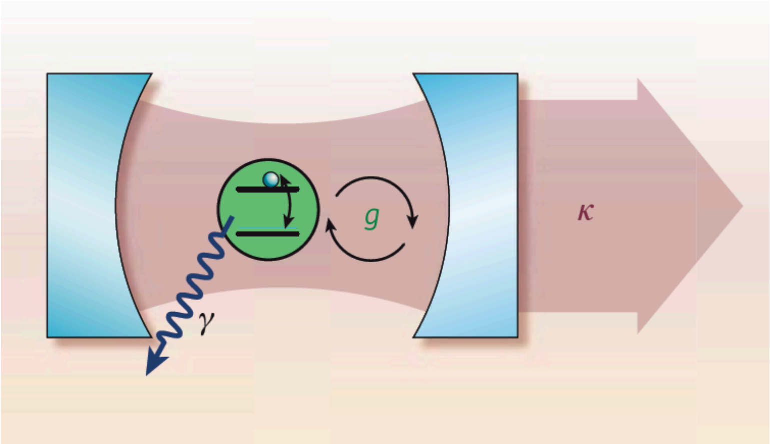
$$H_{JC} = \omega a^\dagger a + \Delta\sigma_z + g(\sigma_- a^\dagger + \sigma_+ a)$$

easily solvable
workhorse model in quantum optics!

# Renewed interest in the quantum Rabi model…

- breakdown of the rotating wave approximation in cavity QED

for a review, see
A. F. Kockum et al., Nat. Rev. Phys. **1**, 19 (2019)




"ultrastrong" coupling: $0.1 \lesssim g/\omega \lesssim 1$

$g > \kappa, \gamma$

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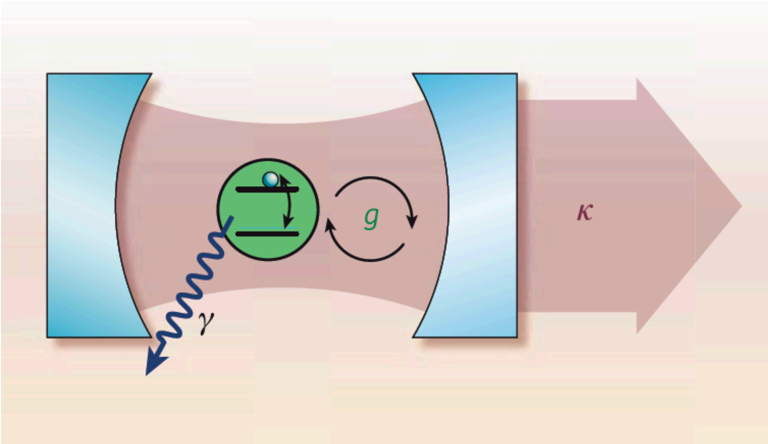
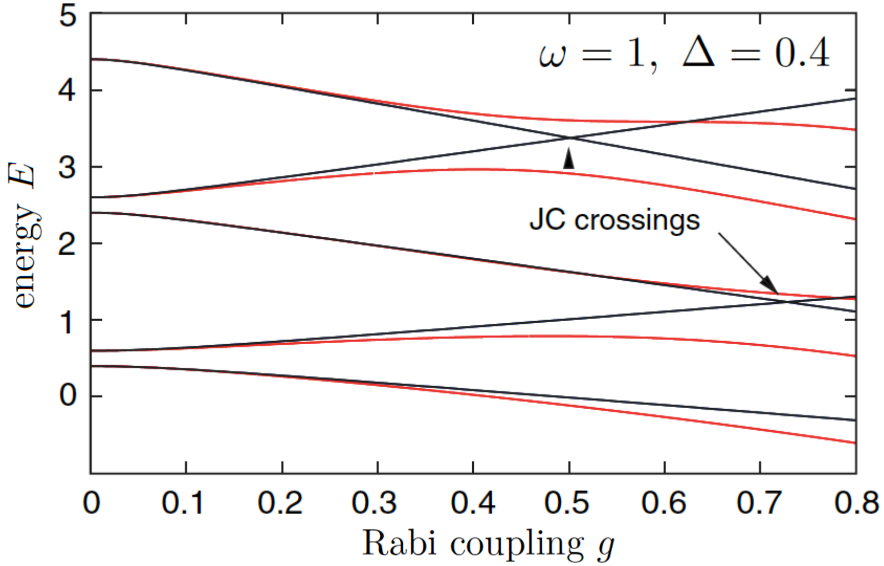



"ultrastrong" coupling: $0.1 \lesssim g/\omega \lesssim 1$

$g > \kappa, \gamma$

- exact solvability of the model

D. Braak, Phys. Rev. Lett. **107**, 100401 (2011)



## Recent results on the model (selected)

- *superradiant transition* at a critical Rabi coupling $g_c$ when $\Delta/\omega \to \infty$
  M.-J. Hwang, P. Puebla, and M. B. Plenio, Phys. Rev. Lett. **115**, 180404 (2015)
- nonperturbative effects in the *two-photon* quantum Rabi model
  L. Duan, Y.-F. Xie, D. Braak, and Q.-H. Chen, J. Phys. A: Math. Gen. **49**, 464002 (2016)
- scaling and universality in the *anisotropic* quantum Rabi model
  M. Liu, S. Chesi, Z.-J. Ying, X. Chen, H.-G. Luo, and H.-Q. Lin, Phys. Rev. Lett. **119**, 220601 (2017)
- polaron picture in the *two-qubit* quantum Rabi model
  X.-M. Sun, L. Cong, H.-P. Eckle, Z.-J. Ying, and H.-G. Luo, Phys. Rev. A **101**, 063832 (2020)
- …
- …
- and many more …

# Quantum Rabi-Stark model

A. L. Grimsmo and S. Parkins, Phys. Rev. A **87**, 033814 (2013)

$$H_{RS} = \omega a^\dagger a + \sigma_z(\Delta + \gamma a^\dagger a) + g\sigma_x(a^\dagger + a)$$

*Stark coupling*

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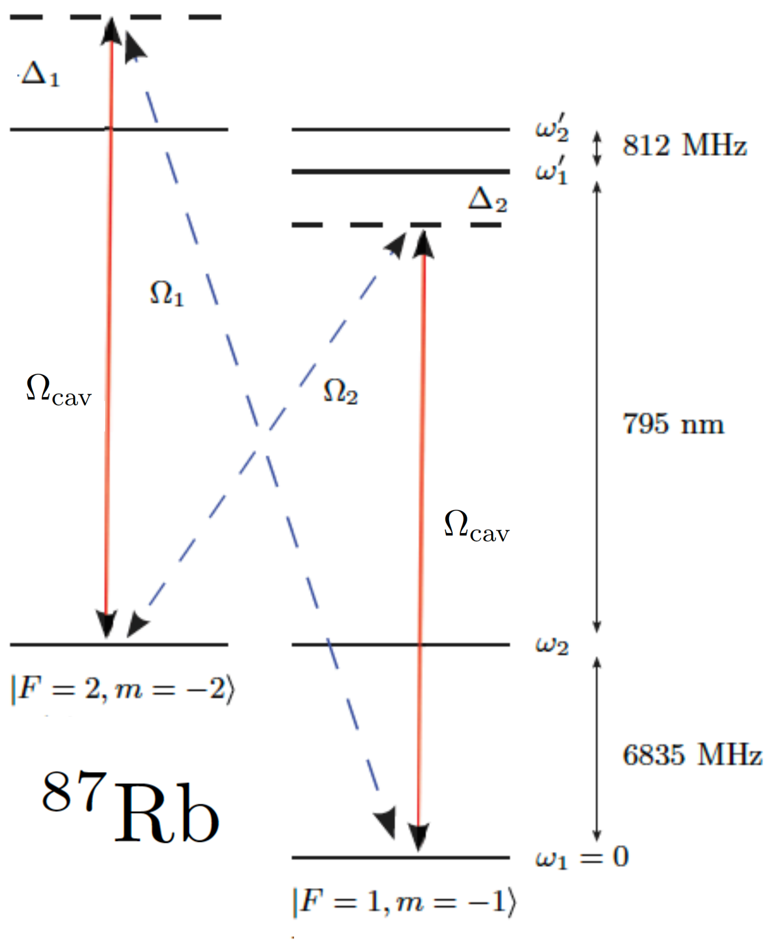
**physical motivation:**
tunable platform for cavity QED
in ultrastrong coupling regime




$\Omega_{1,2}$ *laser frequencies*

$\Omega_{\rm cav}$ *cavity mode frequency*

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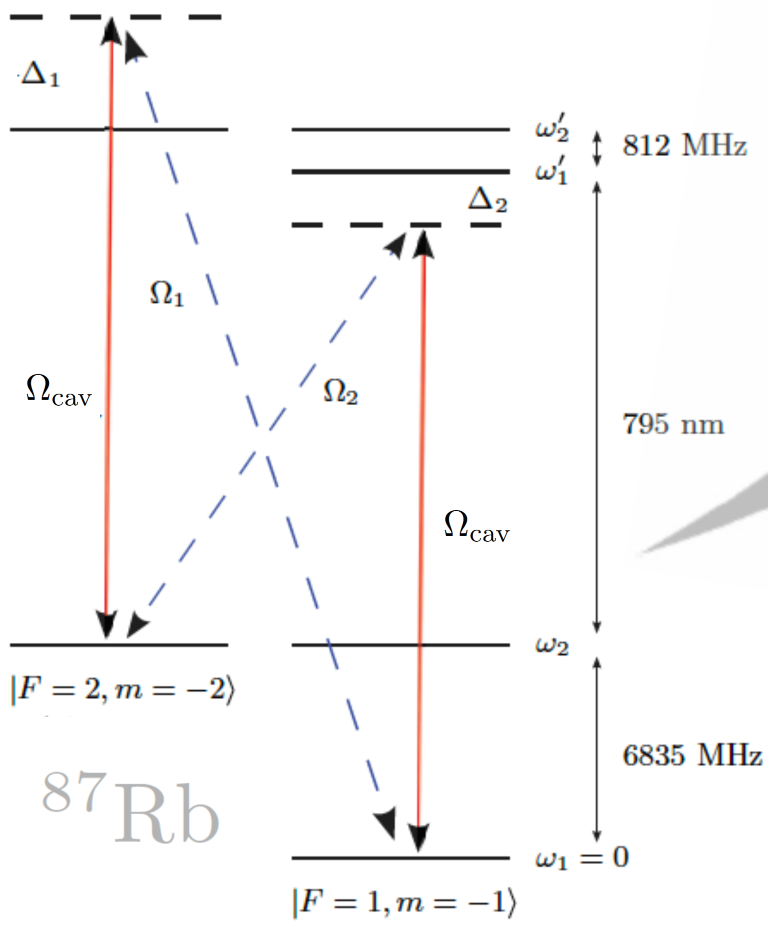


**effective model**
from integrating out the virtual levels – – – –

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$$H_{RS} = \omega a^\dagger a + \sigma_z(\Delta + \gamma a^\dagger a) + g\sigma_x(a^\dagger + a)$$

independently tunable via laser frequencies and intensities

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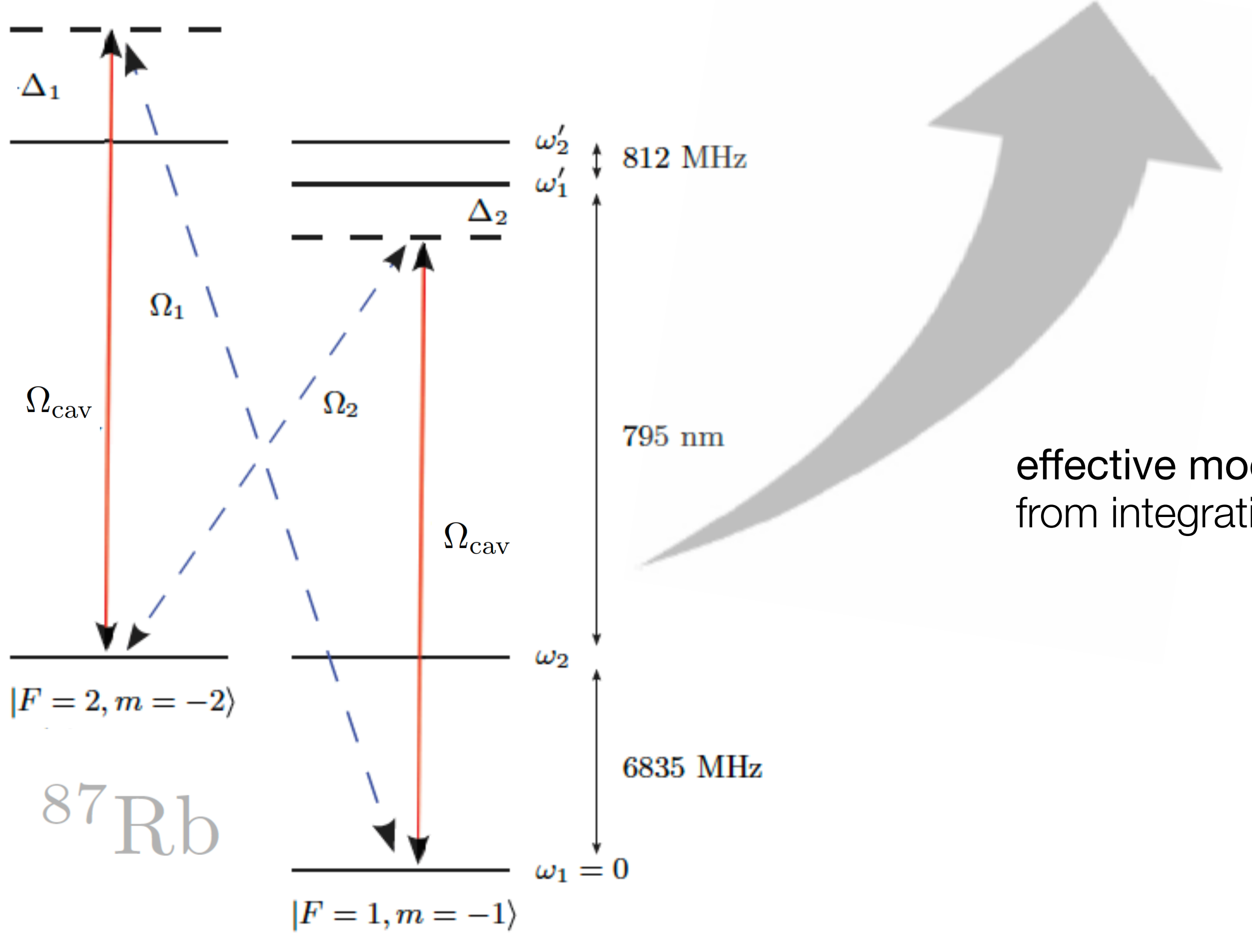


effective model
from integrating out the virtual levels

# Spectrum of the quantum Rabi-Stark model…

$$H_{RS} = \omega a^\dagger a + \sigma_z(\Delta + \gamma a^\dagger a) + g\sigma_x(a^\dagger + a)$$

same $\mathbb{Z}_2$ parity symmetry as for Rabi model

$$\mathcal{P} = -\sigma_z \otimes (-1)^{a^\dagger a}$$

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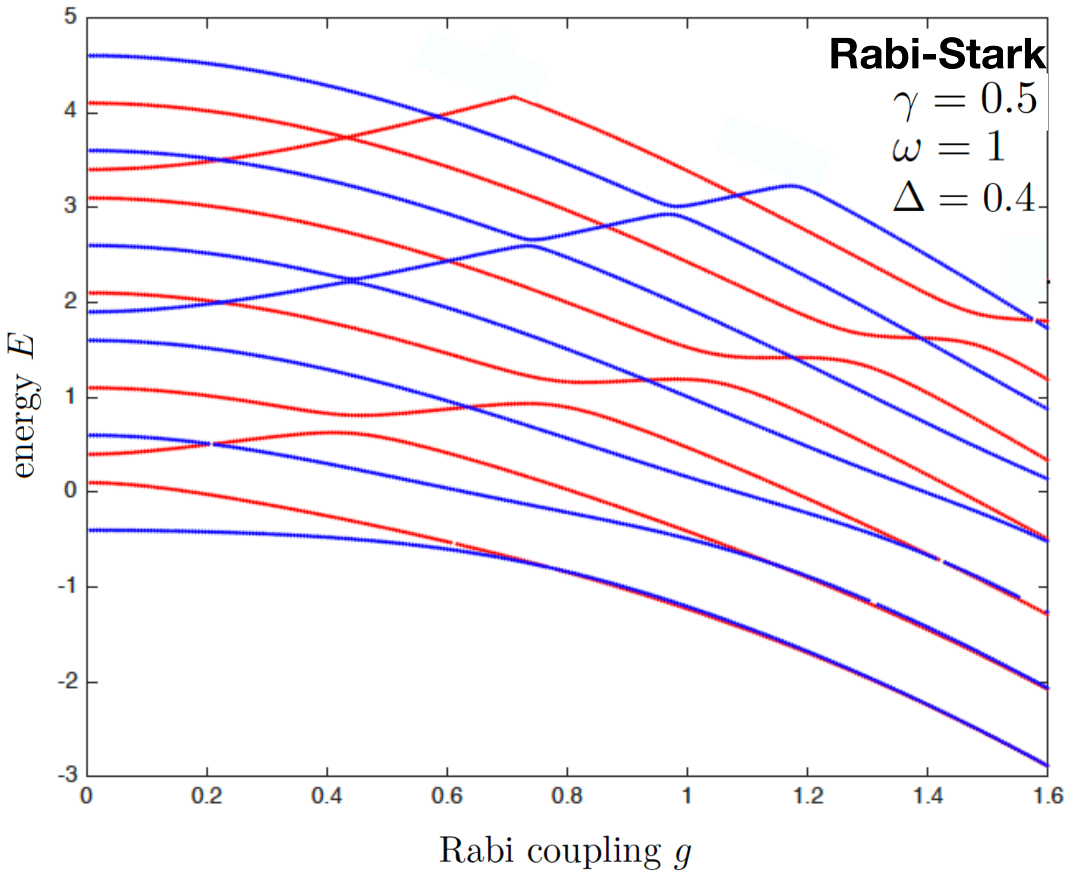


H.-P. Eckle and HJ, J. Phys. A: Math. Theor. **50**, 294004 (2017); erratum: ibid. **56**, 345302 (2023)

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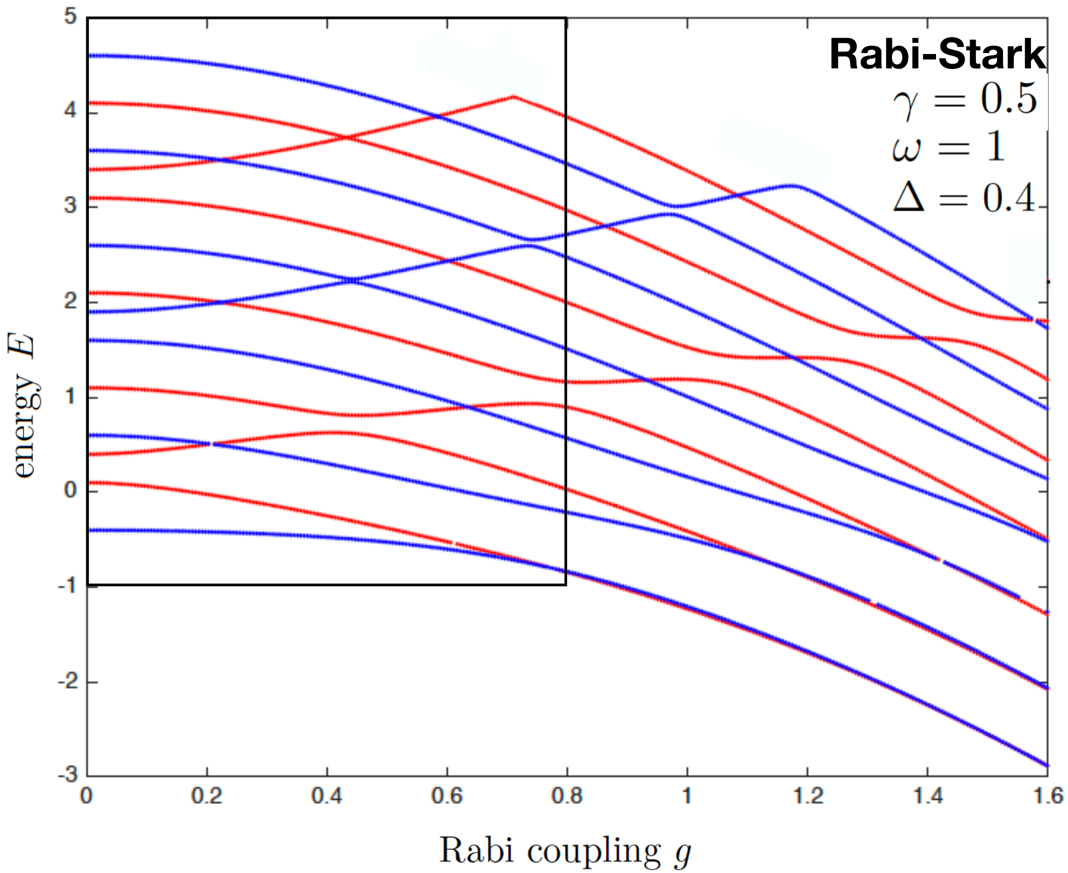


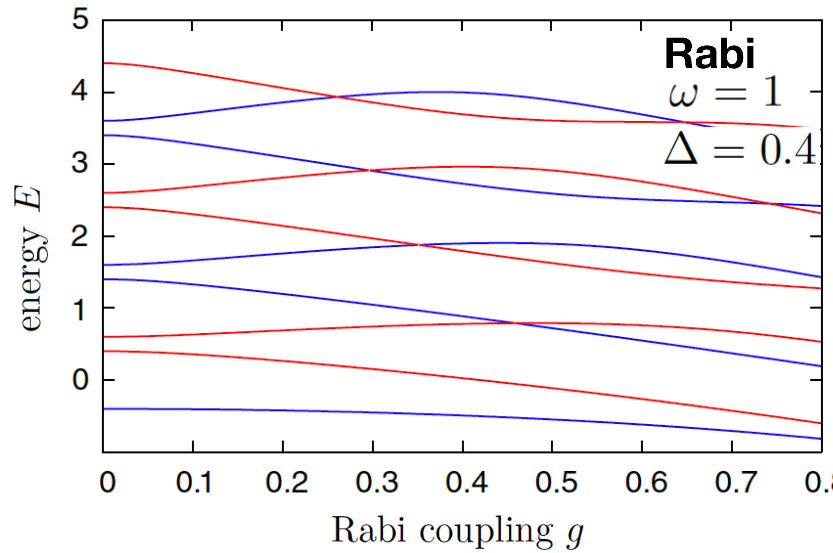


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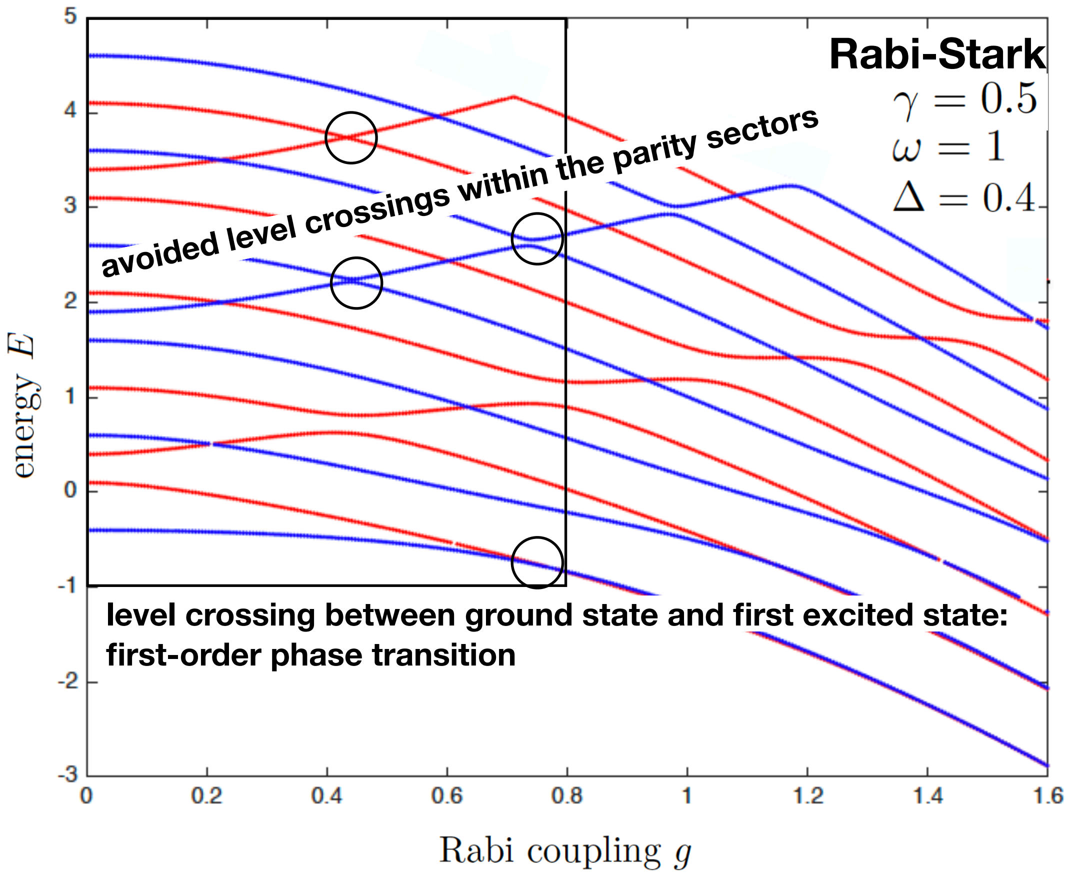


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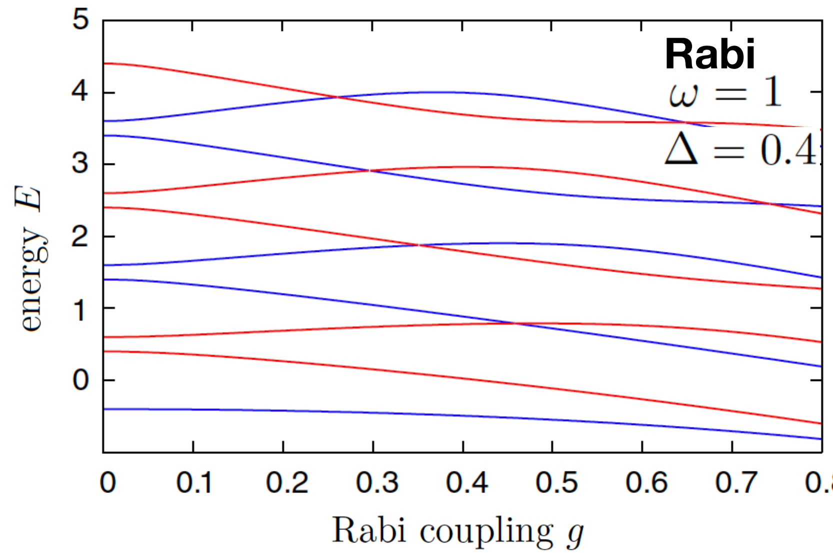


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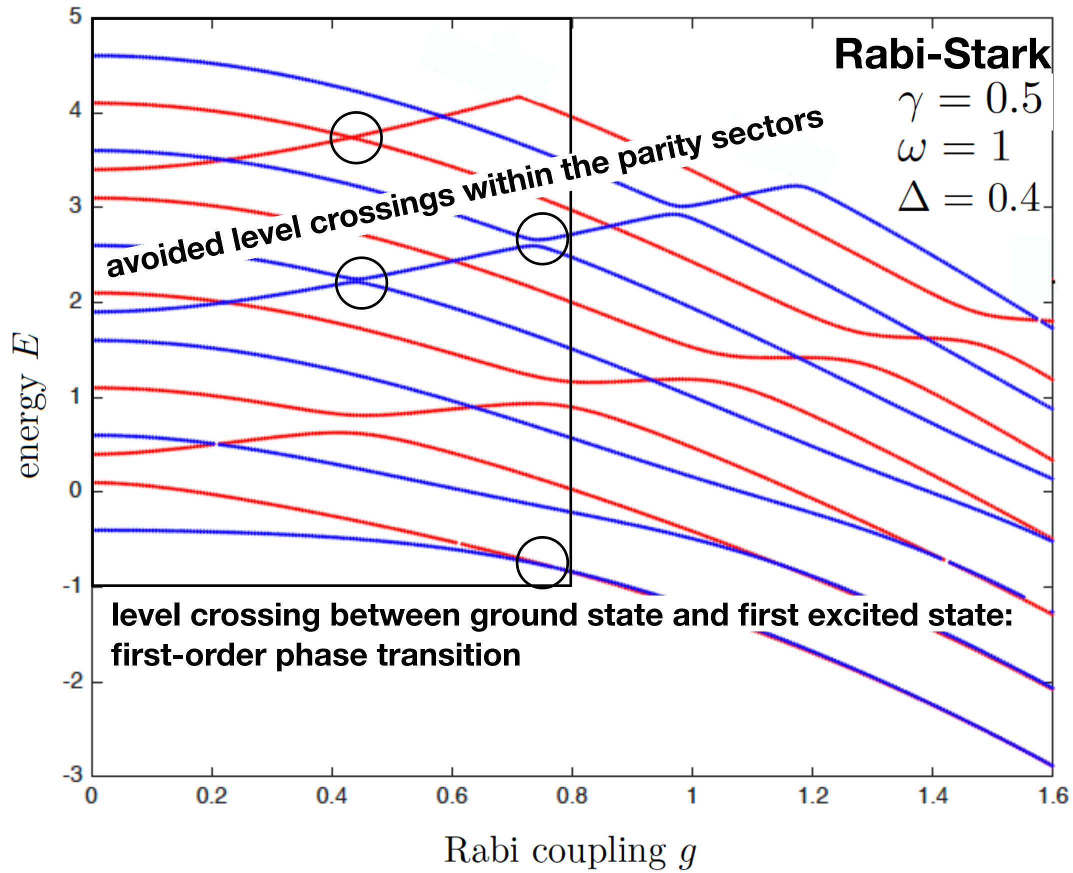


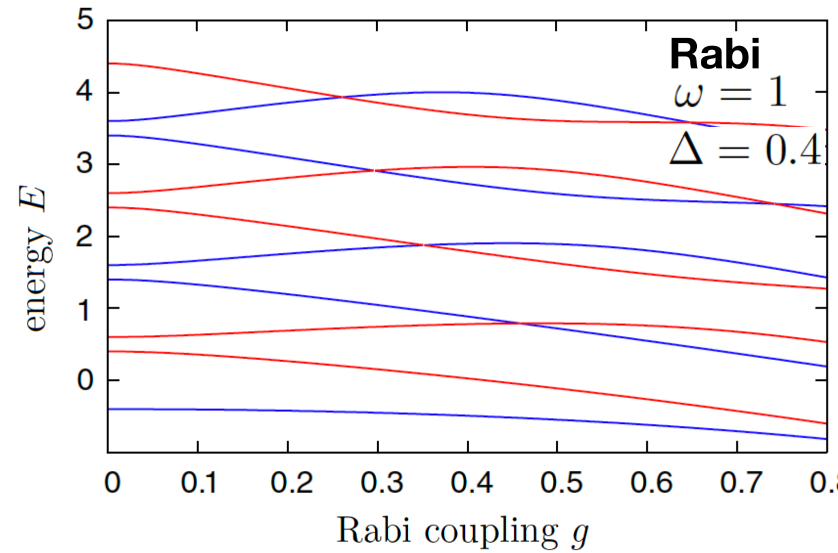


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**N. B. Solution obtainable only for $|\gamma| < \omega$. What happens when $|\gamma| > \omega$ ?**

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Finite-range **spectral continuum** ***or*** **gap** above the ground state

Y.-F. Xie, L. Duan, and Q.-H. Chen, J. Phys. A: Math. Theor. **52**, 245304 (2019)

# Spectral collapse, continuum, or gap?

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Strategy: explore the limit $\gamma \to \omega_-$ (taking $\gamma > 0$)

# Spectral collapse, continuum, or gap?

## Preliminaries

Bargmann representation

V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961)

Bargmann space

$$a^\dagger \to z, \quad a \to \frac{\mathrm{d}}{\mathrm{d}z}, \qquad |\psi\rangle \to \psi(z) \in \mathcal{B}$$

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$$a^\dagger \to z, \quad a \to \frac{\mathrm{d}}{\mathrm{d}z}, \qquad |\psi\rangle \to \boxed{\psi(z) \in \mathcal{B}}$$

- $\psi(z)$ entire
- $\langle\psi|\psi\rangle < \infty$

$$\langle\psi|\phi\rangle = \frac{1}{\pi}\int \mathrm{d}z\mathrm{d}\bar{z}\,\overline{\psi(z)}\phi(z)e^{-z\bar{z}}$$

Schrödinger equation

$$H_{RS}\begin{pmatrix}|\phi_1\rangle\\|\phi_2\rangle\end{pmatrix}=E\begin{pmatrix}|\phi_1\rangle\\|\phi_2\rangle\end{pmatrix}$$

Bargmann

$$\begin{aligned}(\omega+\gamma)z\frac{\mathrm{d}}{\mathrm{d}z}\phi_1+\Delta\phi_1+g\left(z+\frac{\mathrm{d}}{\mathrm{d}z}\right)\phi_2 &= E\phi_1,\\ (\omega-\gamma)z\frac{\mathrm{d}}{\mathrm{d}z}\phi_2-\Delta\phi_2+g\left(z+\frac{\mathrm{d}}{\mathrm{d}z}\right)\phi_1 &= E\phi_2,\end{aligned}$$

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scale transformation $\phi_1(z)=\tilde{\phi}_1(z), \qquad \phi_2(z)=\eta\tilde{\phi}_2(z), \quad \eta=\frac{\omega+\gamma}{\omega-\gamma}$

rotation $\begin{pmatrix}\tilde{\phi}_1\\\tilde{\phi}_2\end{pmatrix}=\begin{pmatrix}\sqrt{\eta} & \sqrt{\eta}\\-1 & 1\end{pmatrix}\begin{pmatrix}\bar{\phi}_1\\\bar{\phi}_2\end{pmatrix}$

division by $\omega+\gamma$

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division by $\omega+\gamma$

$$\begin{aligned}(z-\tilde{g})\frac{\mathrm{d}}{\mathrm{d}z}\bar{\phi}_1-(\tilde{g}z+\tilde{E})\bar{\phi}_1&=-\tilde{\Delta}\bar{\phi}_2\\(z+\tilde{g})\frac{\mathrm{d}}{\mathrm{d}z}\bar{\phi}_2+(\tilde{g}z-\tilde{E})\bar{\phi}_2&=-\tilde{\Delta}\bar{\phi}_1\end{aligned}$$

$$\tilde{g}=\frac{g}{\sqrt{\omega^2-\gamma^2}},\quad \tilde{E}=\frac{1}{\omega^2-\gamma^2}(\omega E+\gamma\Delta),\quad \tilde{\Delta}=\frac{1}{\omega^2-\gamma^2}(\omega\Delta+\gamma E)$$

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division by $\omega+\gamma$

take $\gamma<\omega$

analyze $\gamma\to\omega_-$

$$(z-\tilde{g})\frac{\mathrm{d}}{\mathrm{d}z}\bar{\phi}_1-(\tilde{g}z+\tilde{E})\bar{\phi}_1 = -\tilde{\Delta}\bar{\phi}_2$$
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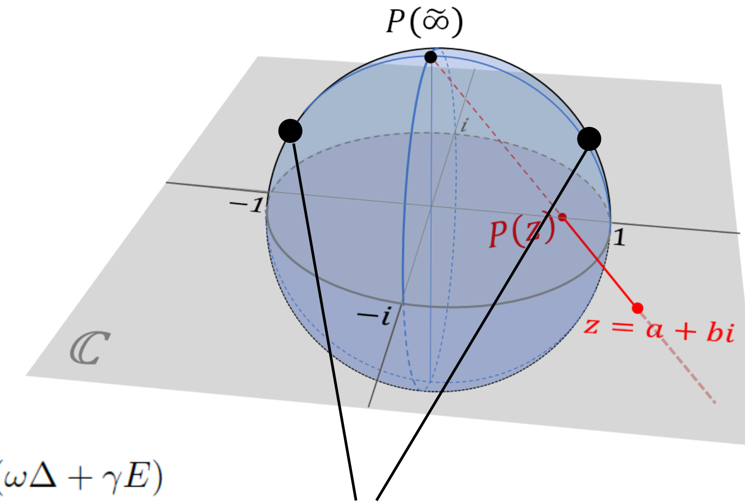


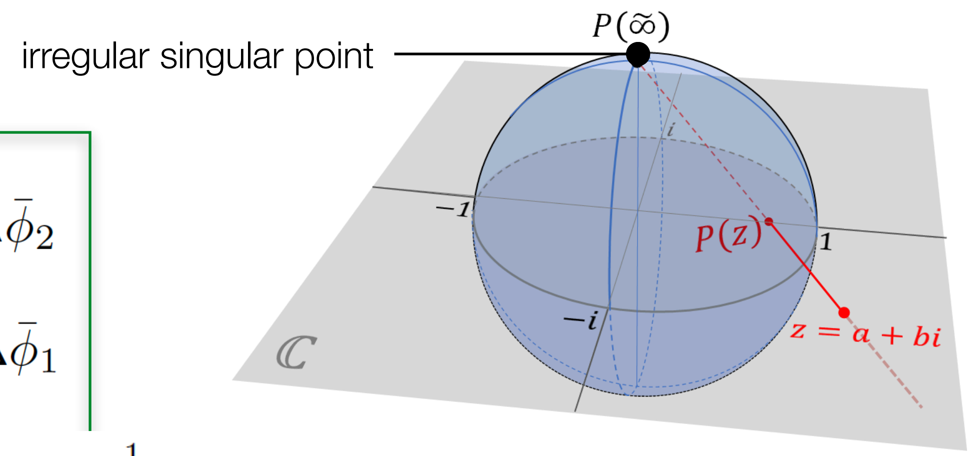

regular singular points at $z=\pm\tilde{g}$

# The limit $\gamma \to \omega_-$

$$(z-\tilde{g})\frac{\mathrm{d}}{\mathrm{d}z}\bar{\phi}_1 - (\tilde{g}z+\tilde{E})\bar{\phi}_1 = -\tilde{\Delta}\bar{\phi}_2$$

$$(z+\tilde{g})\frac{\mathrm{d}}{\mathrm{d}z}\bar{\phi}_2 + (\tilde{g}z-\tilde{E})\bar{\phi}_2 = -\tilde{\Delta}\bar{\phi}_1$$

$$\tilde{g} = \frac{g}{\sqrt{\omega^2-\gamma^2}}, \quad \tilde{E} = \frac{1}{\omega^2-\gamma^2}(\omega E + \gamma\Delta), \quad \tilde{\Delta} = \frac{1}{\omega^2-\gamma^2}(\omega\Delta+\gamma E)$$



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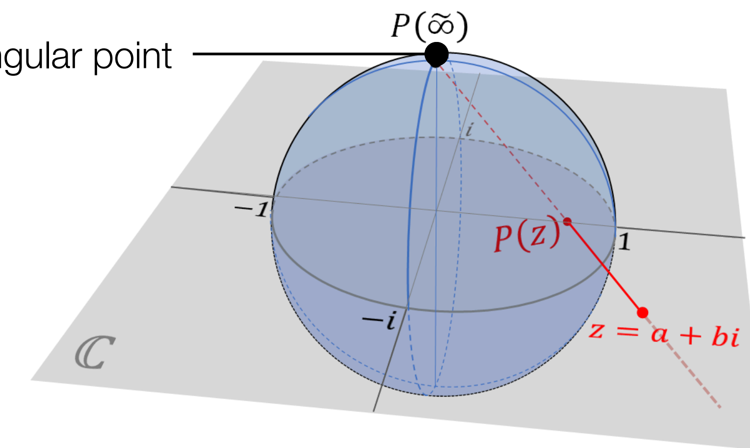


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Confluence Equation
codes for the confluence of the two regular singular points at infinity

derivation: eliminate $\bar{\phi}_2(z)$, take $\gamma \to \omega_-$ and keep the most divergent terms

assume: powers of $z$ multiplying $\bar{\phi}_1(z)$ and its derivatives are bounded w.r.t. the diverging parameters

★ $$\left[\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(1 + \frac{\omega(E+\Delta)}{g^2}\right)\left[z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\right] + \frac{z^2}{2}\right]\bar{\phi}_1 = \Lambda\bar{\phi}_1$$

$$\Lambda = \frac{E^2-\Delta^2}{2g^2} + \frac{\omega}{2g^2}(E+\Delta)$$

# The limit $\gamma \to \omega_-$

$$\star \quad \left[\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(1 + \frac{\omega(E+\Delta)}{g^2}\right)\left[z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\right] + \frac{z^2}{2}\right]\bar{\phi}_1 = \Lambda\bar{\phi}_1$$

$$\Lambda = \frac{E^2 - \Delta^2}{2g^2} + \frac{\omega}{2g^2}(E+\Delta)$$

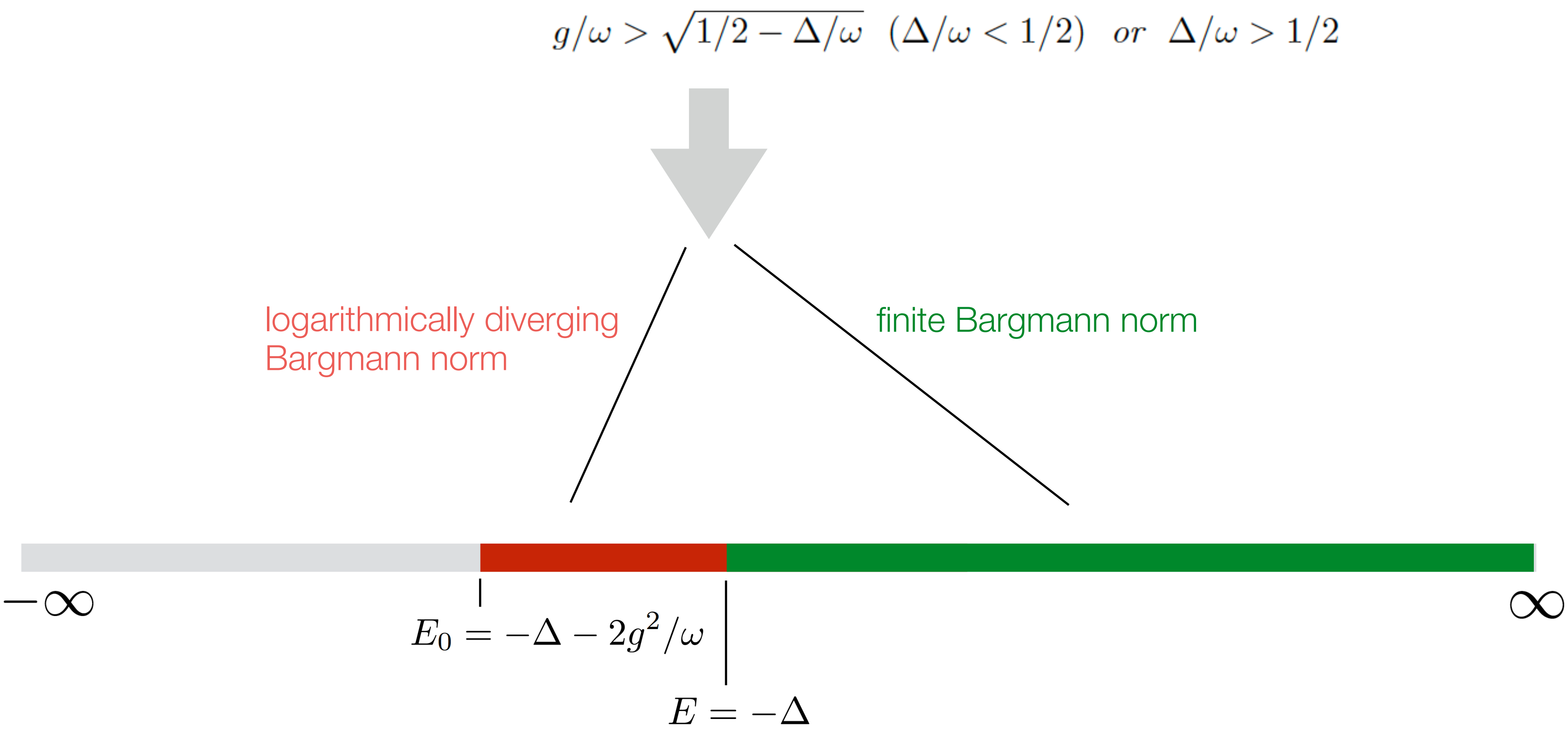
asymptotic solution:

S. Y. Slavyanov and W. Lay, Special Fuctions: A Unified Theory Based on Singularities (OUP, 2000)

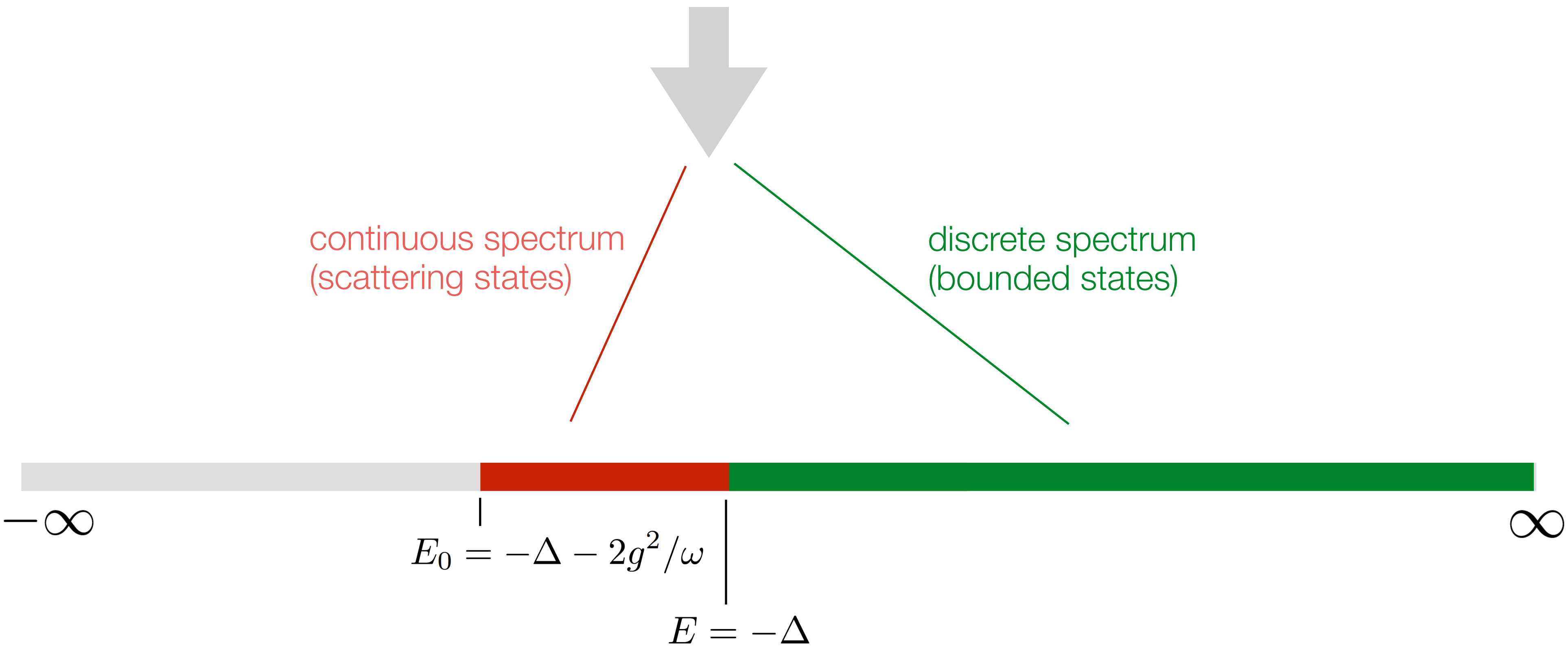
$$\bar{\phi}_1(z) = \exp\left(\frac{\beta_1}{2}z^2 + \beta_2 z\right) z^{\rho} \sum_{n=0}^{\infty} c_n z^{-n}, \qquad z \to \infty$$

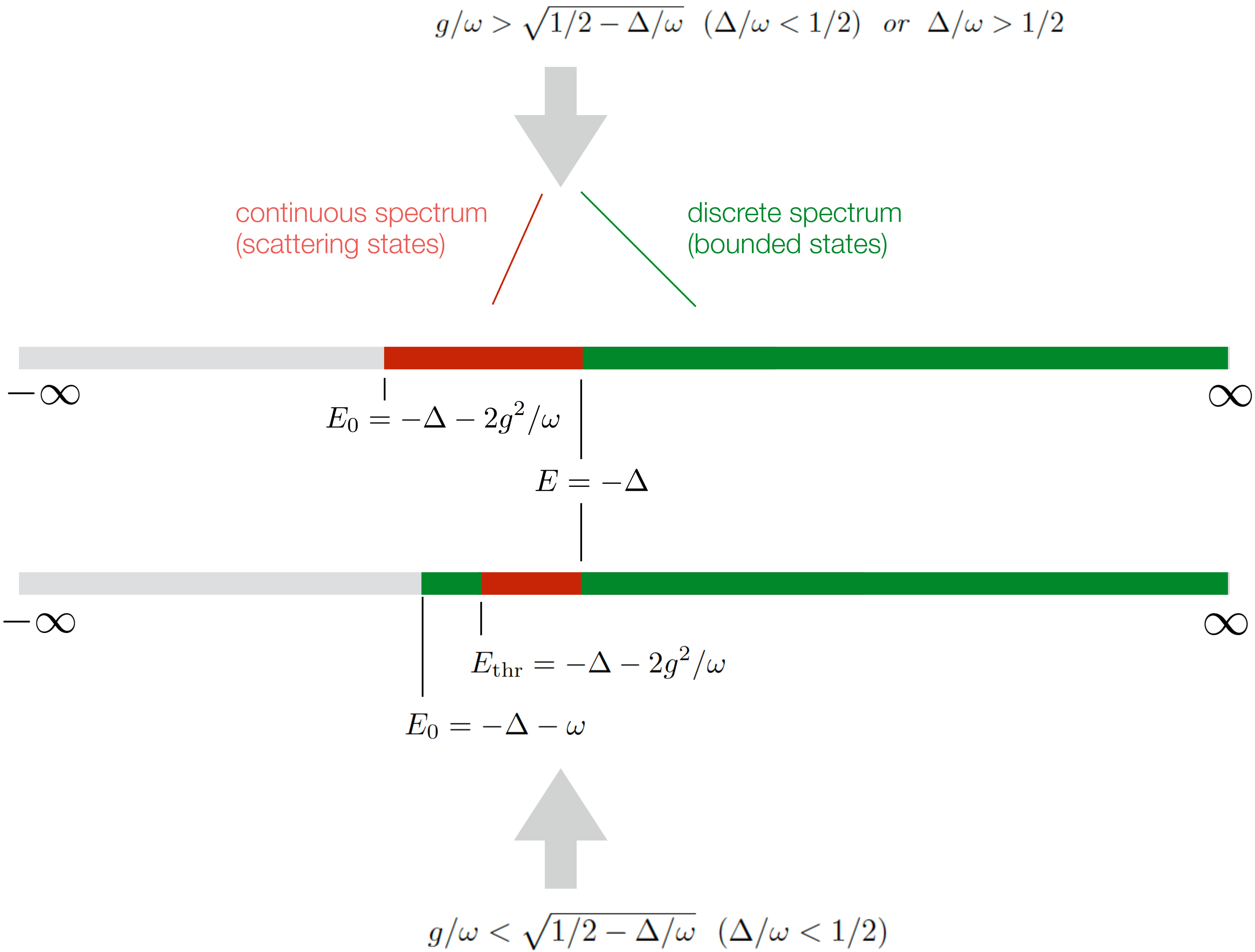
plug into ★ and determine $\beta_1, \beta_2, \rho$

check the Bargmann norm!

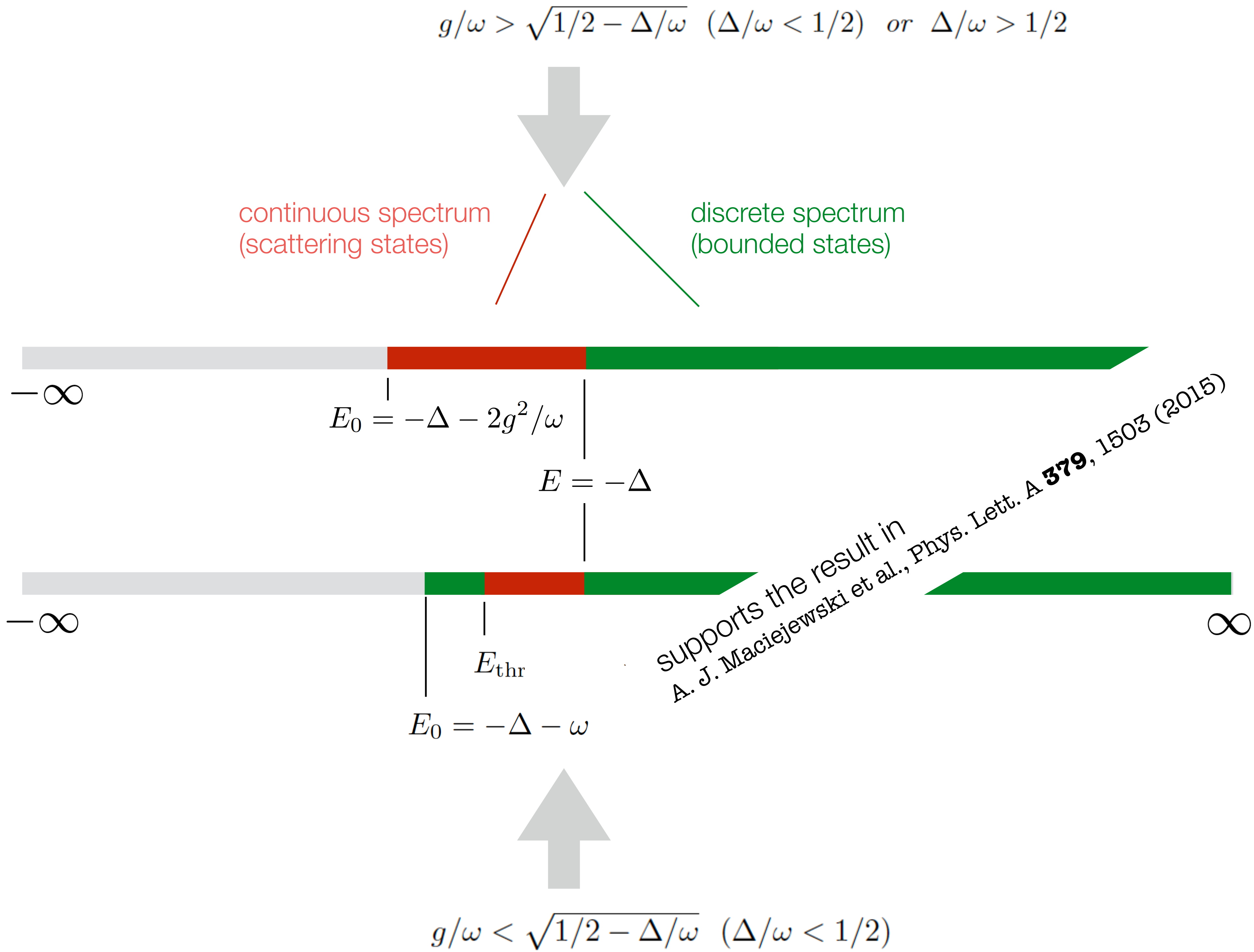

$g/\omega > \sqrt{1/2 - \Delta/\omega}$ $(\Delta/\omega < 1/2)$ $or$ $\Delta/\omega > 1/2$
logarithmically diverging Bargmann norm
finite Bargmann norm
$-\infty$
$\infty$
$E_0 = -\Delta - 2g^2/\omega$
$E = -\Delta$

$$g/\omega > \sqrt{1/2 - \Delta/\omega} \quad (\Delta/\omega < 1/2) \quad or \quad \Delta/\omega > 1/2$$





$g/\omega > \sqrt{1/2 - \Delta/\omega}$ $(\Delta/\omega < 1/2)$ *or* $\Delta/\omega > 1/2$
continuous spectrum
(scattering states)
discrete spectrum
(bounded states)
$-\infty$
$\infty$
$E_0 = -\Delta - 2g^2/\omega$
$E = -\Delta$
$-\infty$
$\infty$
$E_{\mathrm{thr}} = -\Delta - 2g^2/\omega$
$E_0 = -\Delta - \omega$
$g/\omega < \sqrt{1/2 - \Delta/\omega}$ $(\Delta/\omega < 1/2)$



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supports the result in
A. J. Maciejewski et al., Phys. Lett. A **379**, 1503 (2015)
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# Revisiting our analysis

...what about the assumption used for deriving the Confluence Equation?

"...*powers of* $z$ *multiplying* $\bar{\phi}_1(z)$ *and its derivatives are bounded w.r.t. the diverging parameter*"

## Revisiting our analysis

...what about the assumption used for deriving the Confluence Equation?

"...*powers of* $z$ *multiplying* $\bar{\phi}_1(z)$ *and its derivatives are bounded w.r.t. the diverging parameter*"

interpretation

zooming in on the neighborhood of $z/\tilde{g} \sim 0$

## Revisiting our analysis

...what about the assumption used for deriving the Confluence Equation?

~~"...*powers of $z$ multiplying $\bar{\phi}_1(z)$ and its derivatives are bounded w.r.t. the diverging parameters*"~~

additional solutions ?

# Additional solutions?

Bargmann equations

$$\left(\epsilon^2 z \frac{\mathrm{d}}{\mathrm{d}z} - \frac{g\epsilon}{\sqrt{\omega+\gamma}}\left(z + \frac{\mathrm{d}}{\mathrm{d}z}\right) + \Gamma_-\right)\bar{\phi}_1 + \Gamma_+\bar{\phi}_2 \;= 0$$

$$\left(\epsilon^2 z \frac{\mathrm{d}}{\mathrm{d}z} + \frac{g\epsilon}{\sqrt{\omega+\gamma}}\left(z + \frac{\mathrm{d}}{\mathrm{d}z}\right) + \Gamma_-\right)\bar{\phi}_2 + \Gamma_+\bar{\phi}_1 \;= 0$$

$$\Gamma_\pm = \frac{\Delta - E}{2(\omega+\gamma)}\epsilon^2 \pm \frac{\Delta + E}{2} \qquad \epsilon = \sqrt{\omega - \gamma}$$

$$z \to \frac{1}{\sqrt{2}}\left(\zeta\tilde{q} - \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}\tilde{q}}\right), \quad \frac{\mathrm{d}}{\mathrm{d}z} \to \frac{1}{\sqrt{2}}\left(\zeta\tilde{q} + \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}\tilde{q}}\right)$$

$$q = \zeta\tilde{q}$$

take $\gamma \to \omega_-$
with $\zeta = \sqrt{2C}/\epsilon$

$$\left(C\tilde{q}^2 - \sqrt{\frac{2C}{\omega}}g\tilde{q} - \frac{E+\Delta}{2}\right)\bar{\phi}_1 + \frac{E+\Delta}{2}\bar{\phi}_2 \;= 0$$
$$\left(C\tilde{q}^2 + \sqrt{\frac{2C}{\omega}}g\tilde{q} - \frac{E+\Delta}{2}\right)\bar{\phi}_2 + \frac{E+\Delta}{2}\bar{\phi}_1 \;= 0$$

eliminate $\bar{\phi}_2(\tilde{q})$

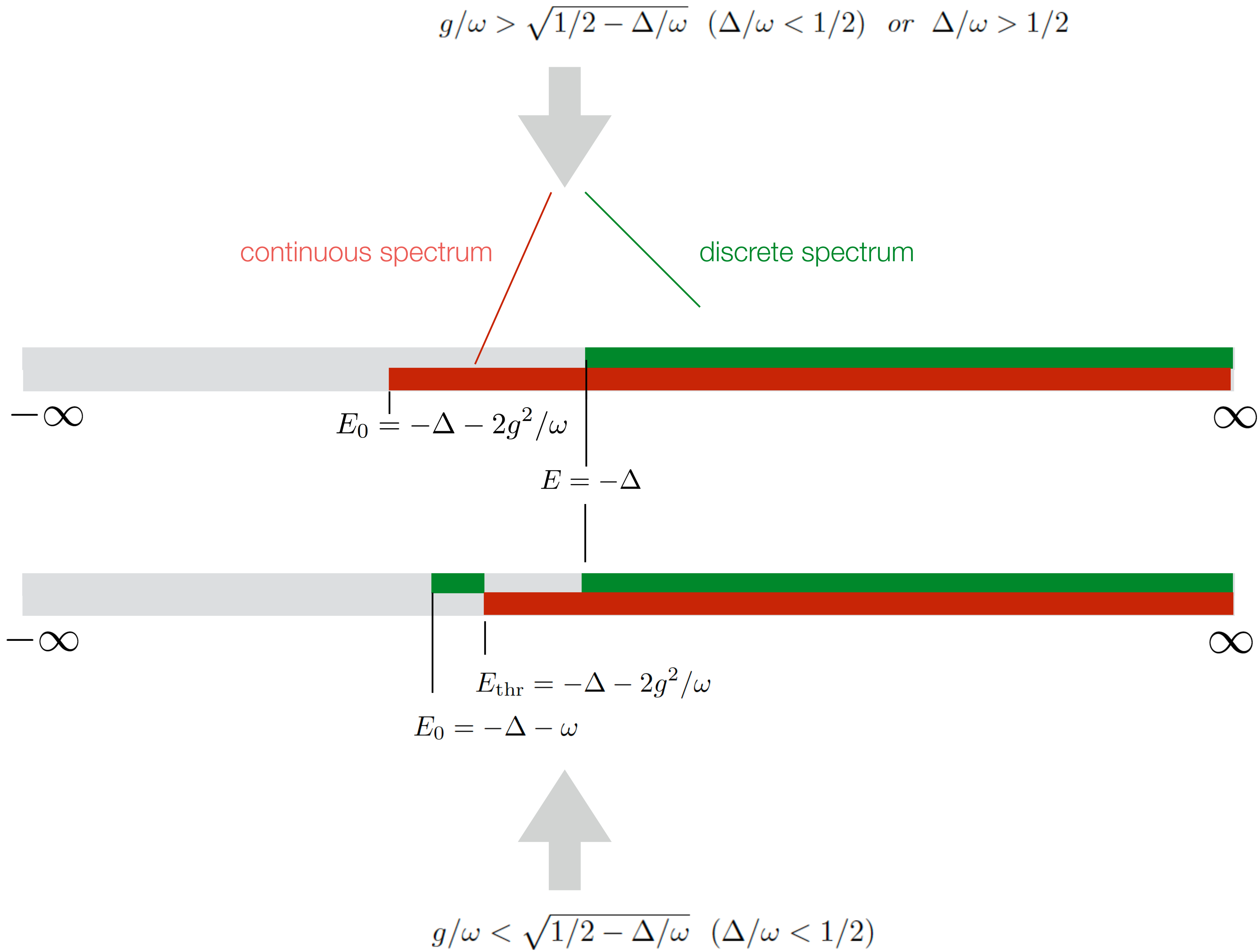
$$\tilde{q}^2\bar{\phi}_1(\tilde{q}) = \frac{1}{C}\left(E + \Delta + \frac{2g^2}{\omega}\right)\bar{\phi}_1(\tilde{q})$$

$\phi_{1,q_0}(q) \propto \delta(q - q_0)$ with $q_0 = \zeta\sqrt{(E - E_{\rm thr})/C}$

generalized eigenfunctions with $E_{\rm thr} = -\Delta - 2g^2/\omega$

$\tilde{q}^2 \geq 0 \longrightarrow E \geq E_{\rm thr}$

spectral continuum of $H_{RS}$ on the whole real axis above $E_{\rm thr}$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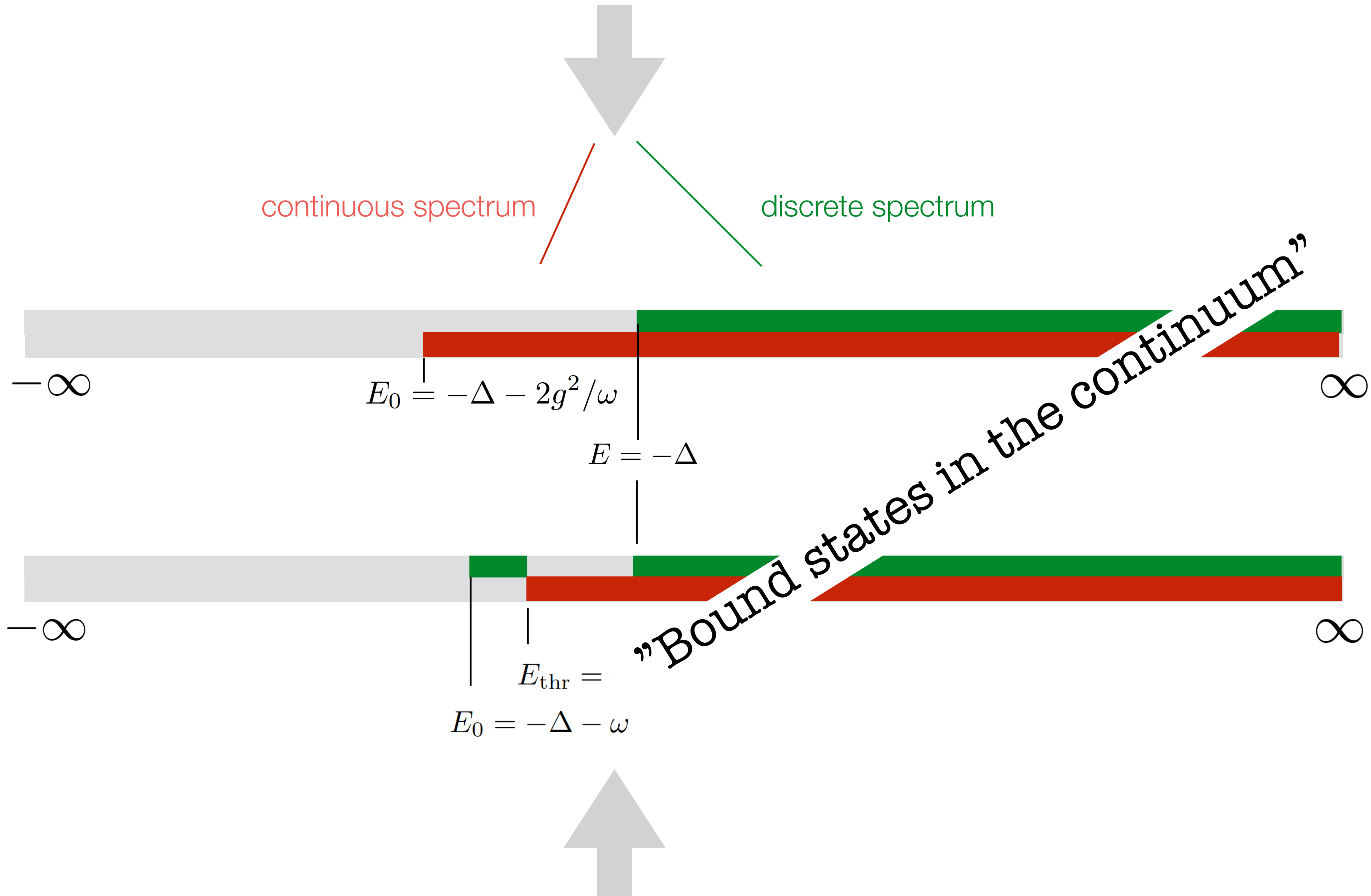
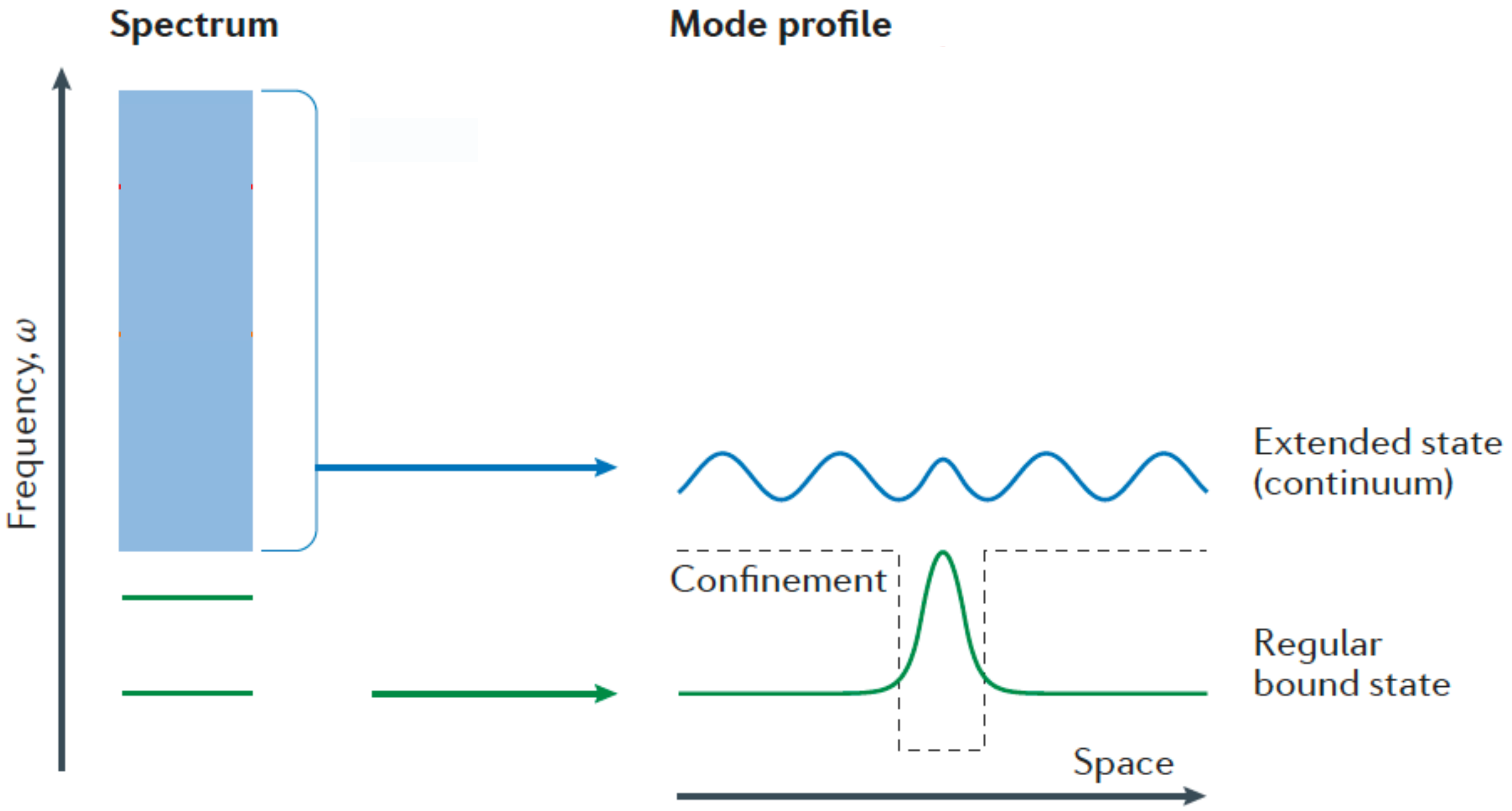



$$g/\omega < \sqrt{1/2 - \Delta/\omega} \quad (\Delta/\omega < 1/2)$$

# Bound states in the continuum (BICs)?



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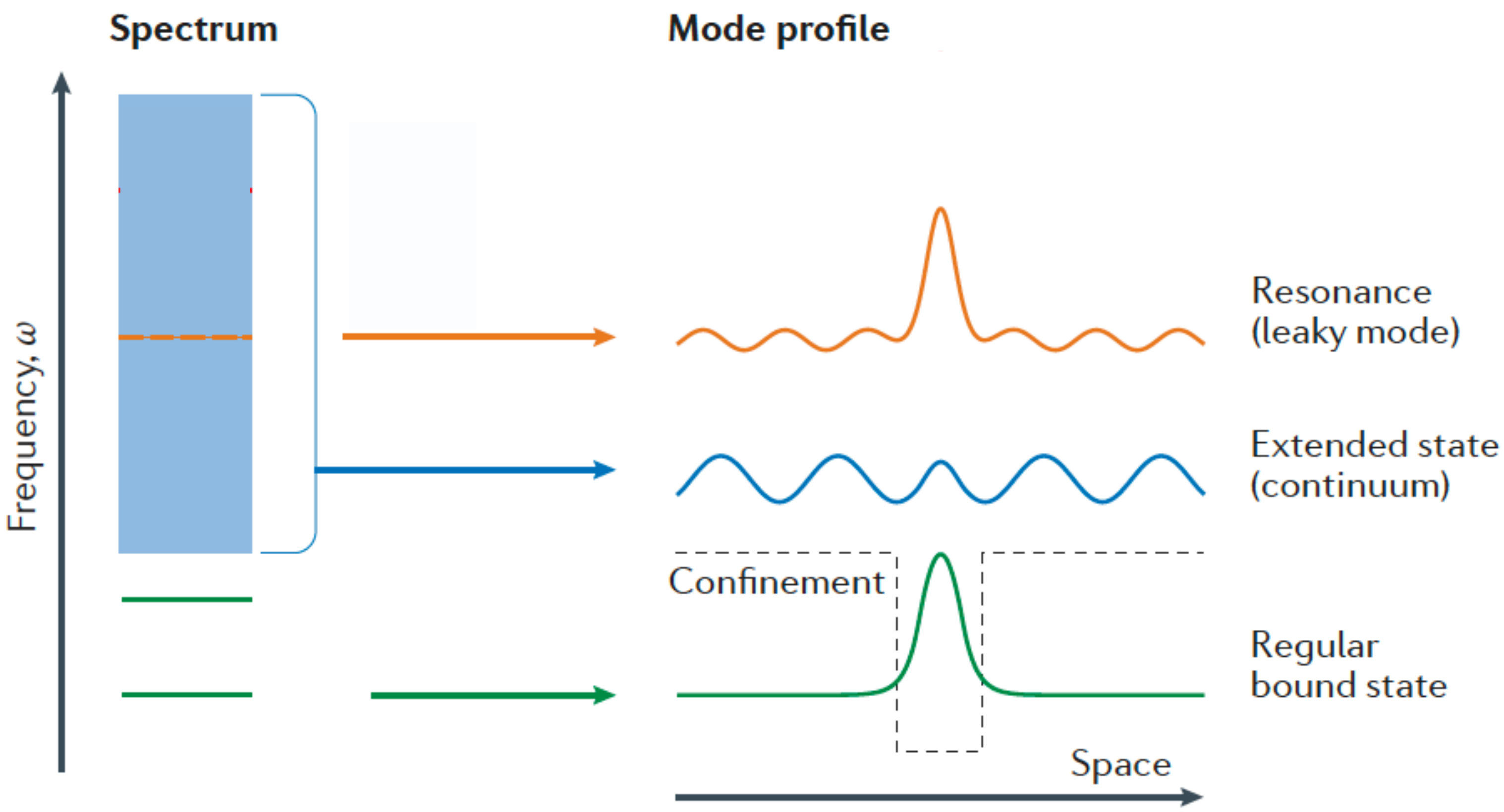
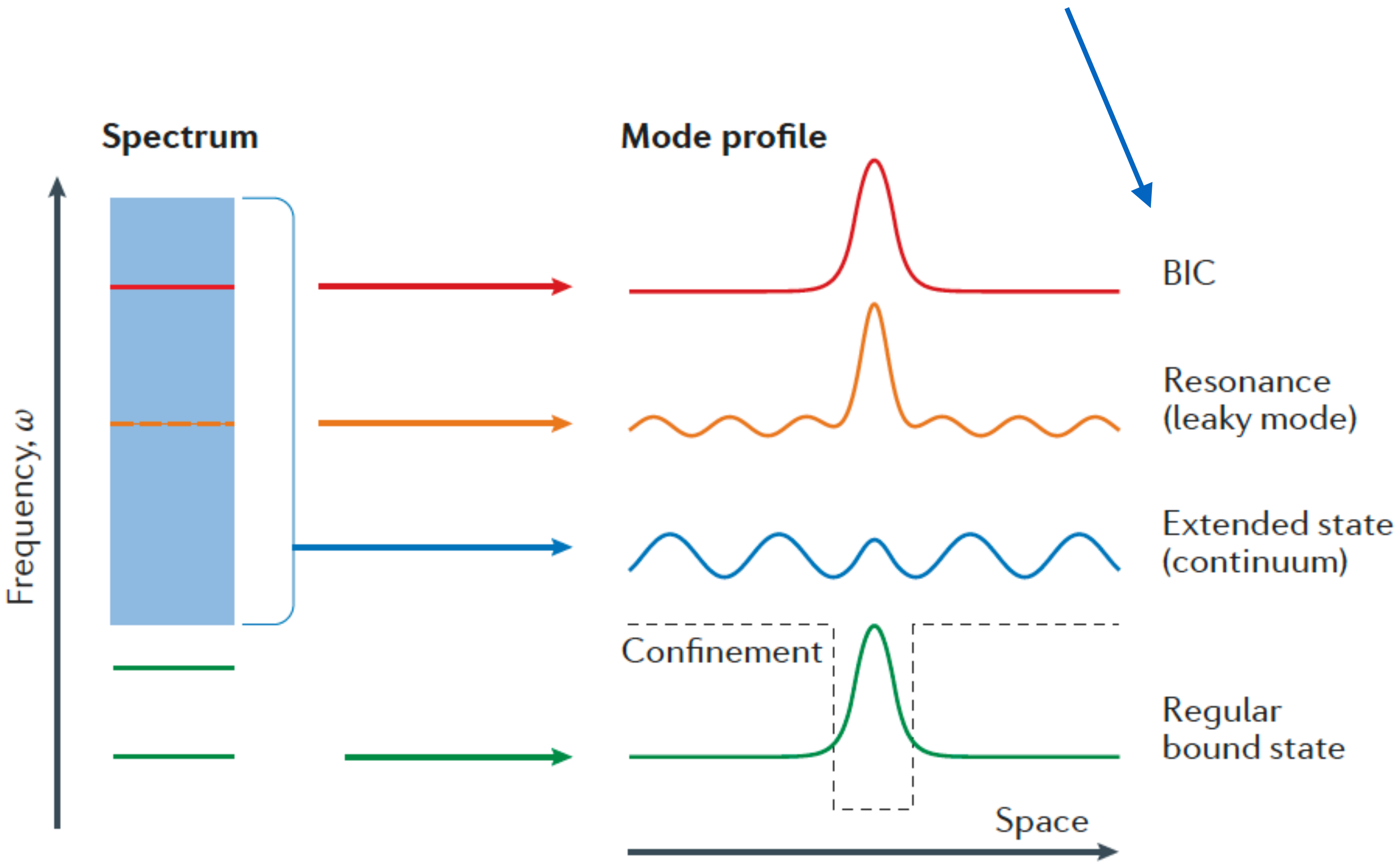


# Bound states in the continuum (BICs)!




from C. W. Chu et al., Nat. Rev. Mater. **1**, 16048 (2016)

# Über merkwürdige diskrete Eigenwerte

J. von Neumann and E. P. Wigner

Physikalische Zeitschrift *30*, 465–467 (1929)

1. Es sind in der quantentheoretischen Literatur mehrfach anschauliche Schlüsse von der Art gemacht worden, daß z. B. aus der Tatsache, daß ein Elektron genügend kinetische Energie hat, um sich aus einem atomaren System (klassisch gerechnet) ins Unendliche zu entfernen, geschlossen wurde, daß der betreffende Energiewert zum kontinuierlichen Spektrum des genannten Systems gehört. Im folgenden soll gezeigt werden, daß derartige Überlegungen mit äußerster Vorsicht zu handhaben sind, denn es kommt häufig ein entgegengesetztes Verhalten vor. Dieser Umstand, daß ein Elektron auf einer stationären Bahn verharrt (Punkteigenwert!), obwohl es Energie genug hätte, um sich aus dem Anziehungsbereich des ihn umgebenden Systems zu befreien, ist nur scheinbar paradox. Wir werden uns an zwei verschiedenen Beispielen klar machen, daß dieses Phänomen zwei ganz verschiedene Ursachen haben kann — aber in beiden Fällen bis zu einem gewissen Grade anschaulich deutbar ist.

Wir werden stets ein Elektron im Bereiche eines geeigneten kugelsymmetrischen Potentials betrachten, also die Wellengleichung

$$-\frac{h^2}{8\pi^2 m}\Delta\psi + (V(r) - E)\psi = 0$$

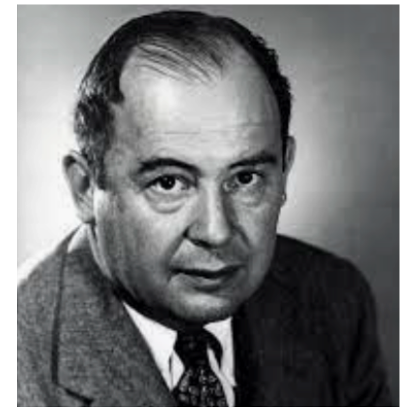
$\left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right.$, da wir uns auf kugelsymmetrische $\psi$ beschränken und die Ableitungen nach den Winkeln $\theta$ und $\varphi$ weglassen können). Damit $E = 0$ ein Punkteigenwert mit der kugelsymmetrischen Eigenfunktion $\psi = \psi(r)$ sei, muß also $V = \frac{h^2}{8\pi^2 m}\left(\frac{\psi''}{\psi} + \frac{2\psi'}{r\psi}\right)$ sein. Wir werden nun durch geeignete Wahl von $\psi$ dem $V$ die oben erwähnten, scheinbar paradoxen, Formen erteilen.

Sei zuerst $\psi = \frac{\sin(r^3)}{r^2}$, dann ergibt sich[1]) (unter Weglassung des unwesentlichen Faktors $\left.\frac{h^2}{8\pi^2 m}\right)$ $V = 2r^{-2} - 9r^4$. Ein ebener Schnitt durch den Potentialverlauf sieht also wie auf der Figur angedeutet aus. Nach allem Erwarten müßte das Elektron den Potentialabhang hinabstürzen und daher nur ein Streckenspektrum besitzen — dennoch ist eine stationäre Bahn und der Punkteigenwert $E = 0$ da!

Jedoch ist dieses Verhalten sogar auf Grund klassisch-mechanischer Analogien zu verstehen[2]).

1) Man setze versuchsweise $\psi(r) = r^a \sin(r^b)$. Damit das Quadratintegral von $\psi$, d.h. $\int_0^\infty 4\pi r^2 |\psi(r)|^2 dr = \int_0^\infty 4\pi r^{a+2}\sin^2(r^b)\,dr$ endlich sei, muß mit Rücksicht auf das Verhalten bei $r = 0$: $2a + b + 2 > -1$ und auf das Verhalten bei $r = \infty$: $2a + 2 < -1$ sein. Die Regularität von $V(r)$ für $r \neq 0$ erfordert $2a + b + 1 = 0$. Beides erfüllt $a = -2$, $b = 3$.

2) Diese Bemerkung rührt von Herrn L. Szilard her.



J. von Neumann

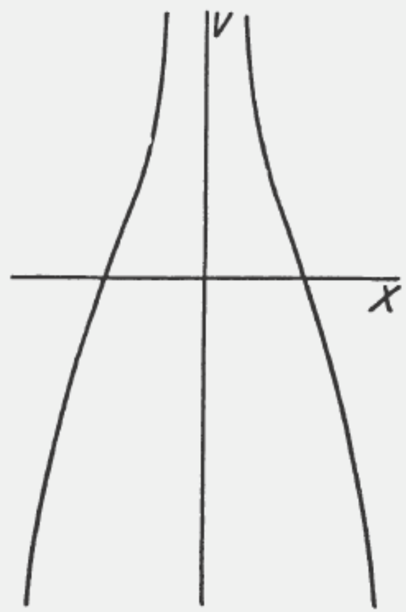
E. P. Wigner

# Bound states in the continuum (BICs)

## theory proposals

- specially tailored potential

J. von Neumann and E. P. Wigner, Phys. Z. **30**, 465 (1929)

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- separable Hamiltonians

  M. Robnik, J. Phys. A: Math. Gen. **19**, 3845 (1986)

example: 2D system

$$H = H_x(x) + H_y(y)$$

$$H_x(x)\psi_x^{(n)}(x) = E_x^{(n)}\psi_x^{(n)}(x)$$

$$H_y(y)\psi_y^{(n)}(y) = E_y^{(n)}\psi_y^{(n)}(y)$$

assume $\psi_x^{(n)}(x)$ and $\psi_y^{(n)}(y)$ are bounded:

$\psi_x^{(n)}(x) \times \psi_y^{(n)}(y)$ will remain bounded even if $E_x^{(n)} + E_y^{(n)}$ lies within the continuous spectrum of $H$

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- symmetry protection

  J. M. Zhang, D. Braak, and M. Kollar, Phys. Rev. Lett. **109**, 116405 (2012)

bound states of one symmetry class embedded in the continuum of states of another symmetry class
$\rightarrow$ no coupling between the states as long as the symmetry is preserved

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- multi-parameter fine tuning

  S. Fan, W. Suh, and J. D. Joannopoulos, J. Opt. Soc. Am. A **20**, 569 (2003)

> blocking leakage of resonances into the continuum by fine tuning parameters (causing destructive interference of the "leaking" channels)

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- one-parameter fine tuning: Rabi-Stark model at $\gamma = \omega$

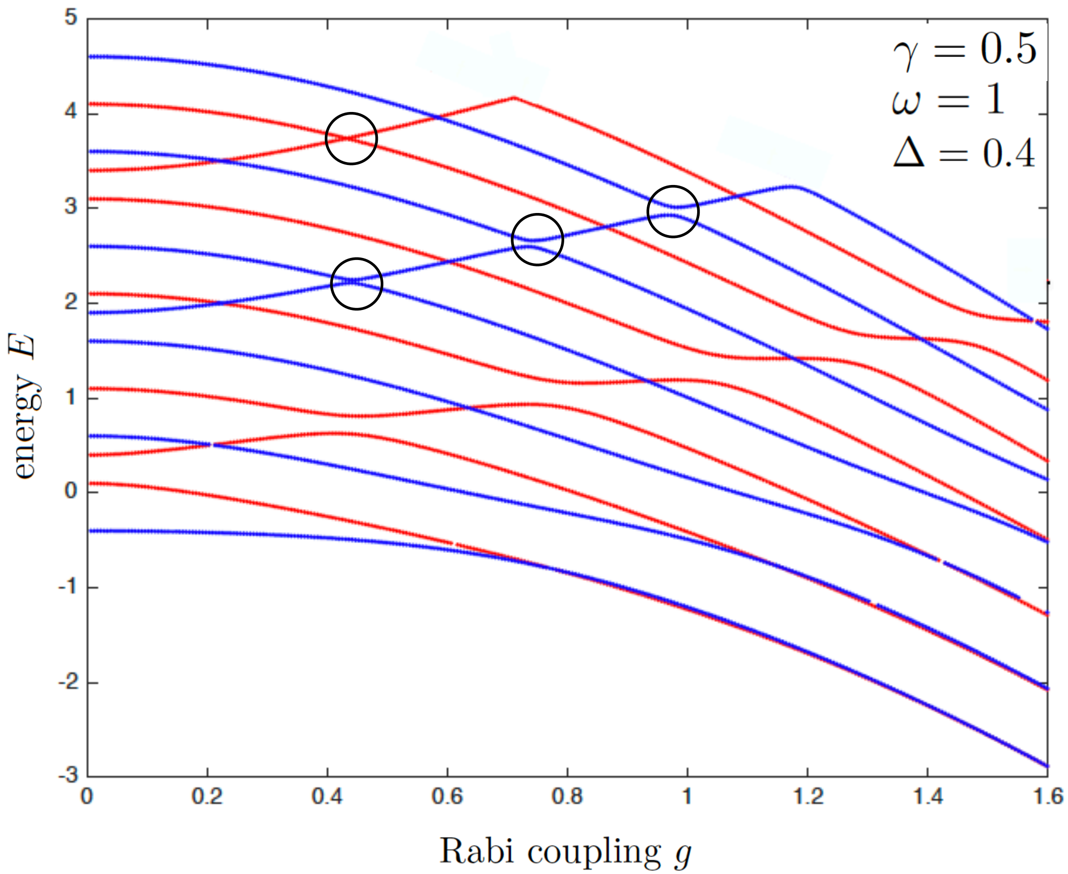
In the neighborhood of $\gamma = \omega$

*preBICS:* precursors of ”true” BICs

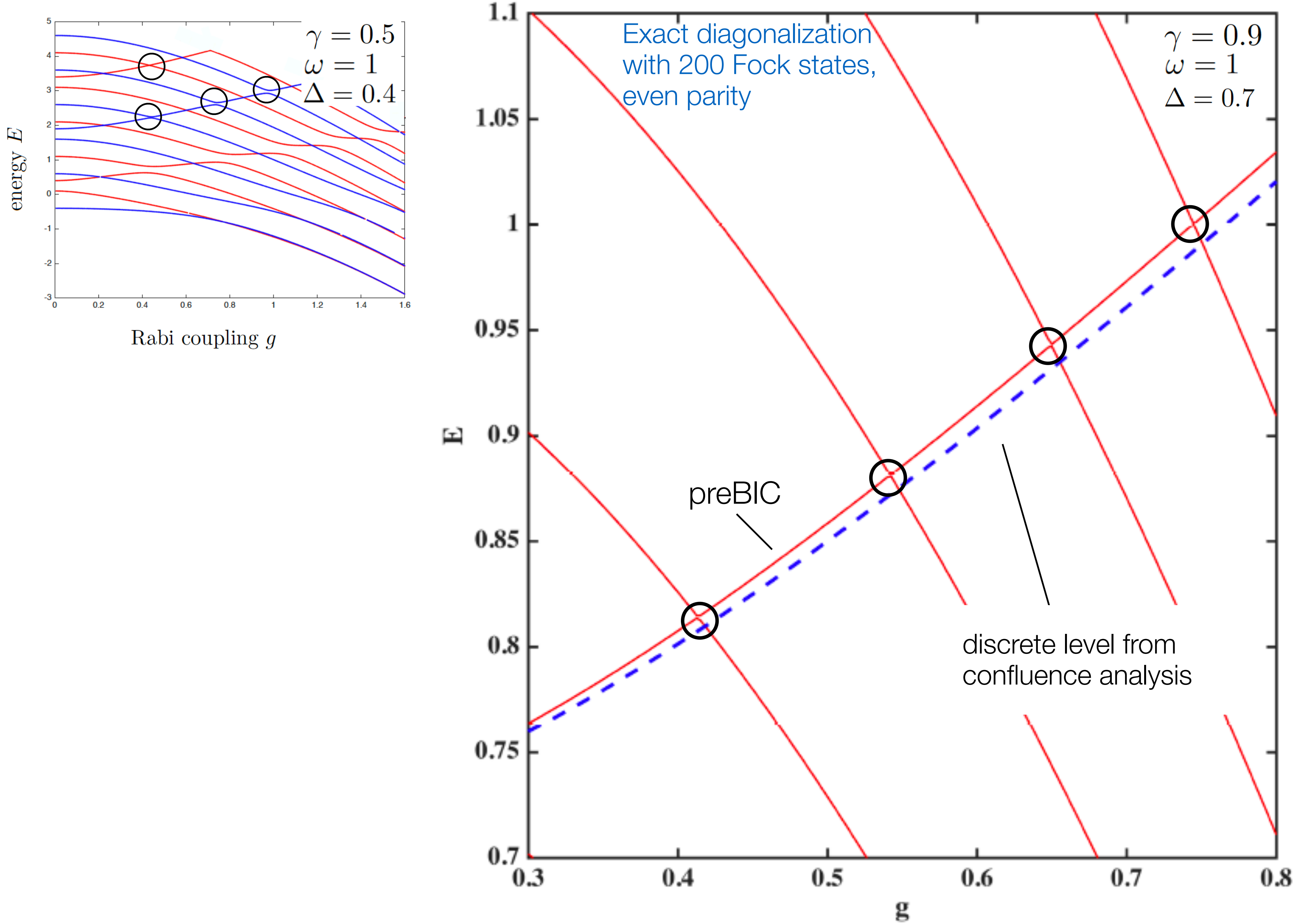
In the neighborhood of $\gamma = \omega$

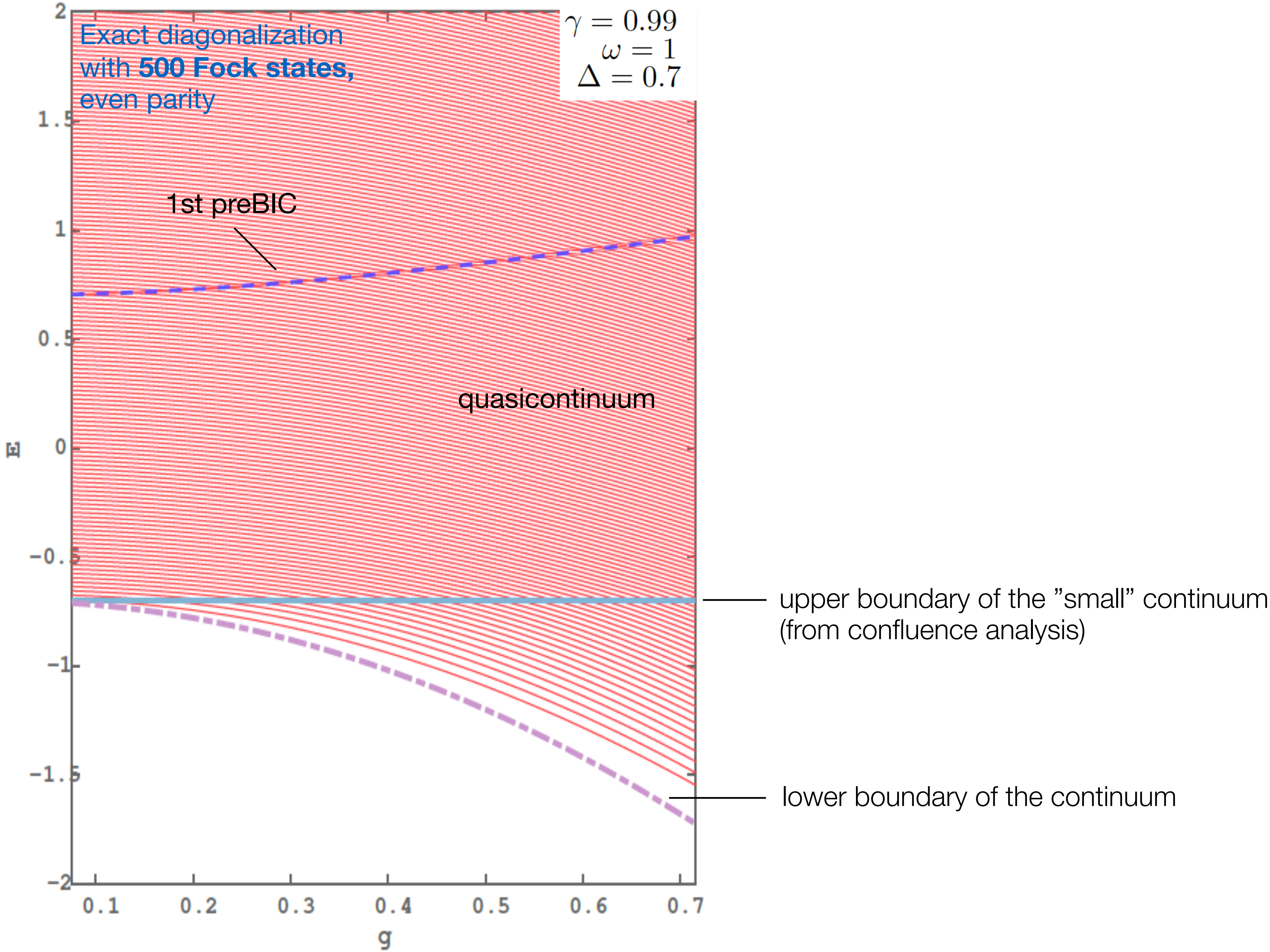
*preBICS:* precursors of ”true” BICs

recall: avoided level crossings (away from $\gamma = \omega$ )

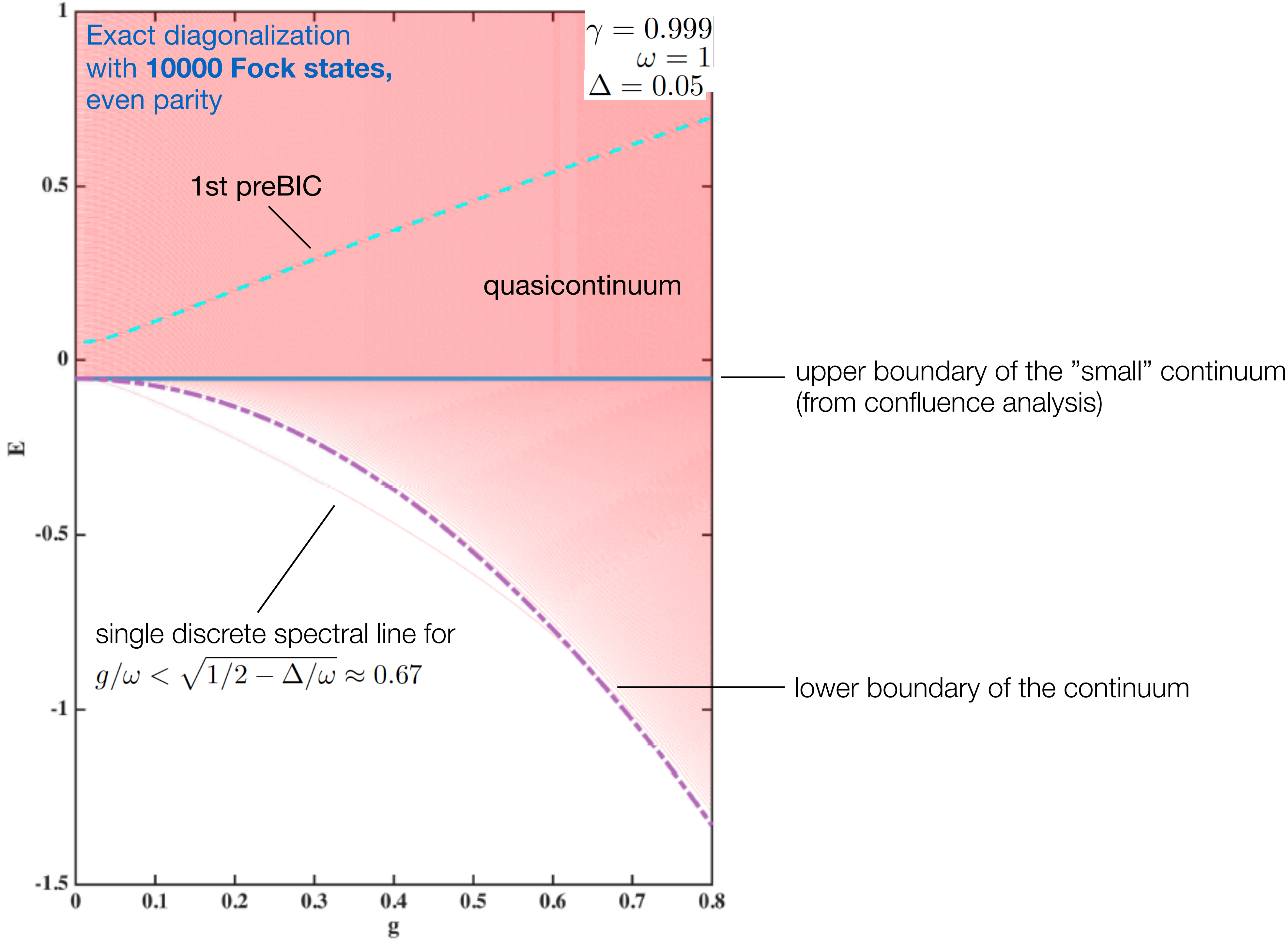



H.-P. Eckle and HJ, J. Phys. A: Math. Theor. **50**, 294004 (2017); erratum: ibid. **56**, 345302 (2023)


γ = 0.5
ω = 1
Δ = 0.4
energy E
Rabi coupling g
Exact diagonalization
with 200 Fock states,
even parity
γ = 0.9
ω = 1
Δ = 0.7
E
g
preBIC
discrete level from
confluence analysis



Exact diagonalization
with **500 Fock states,**
even parity
$\gamma = 0.99$
$\omega = 1$
$\Delta = 0.7$
1st preBIC
quasicontinuum
upper boundary of the "small" continuum
(from confluence analysis)
lower boundary of the continuum
E
g


Exact diagonalization
with **10000 Fock states,**
even parity
$\gamma = 0.999$
$\omega = 1$
$\Delta = 0.05$
1st preBIC
quasicontinuum
upper boundary of the "small" continuum
(from confluence analysis)
single discrete spectral line for
$g/\omega < \sqrt{1/2 - \Delta/\omega} \approx 0.67$
lower boundary of the continuum
E
g
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Emergence of the continuum at $\gamma = \omega$

## ”Slow-mode” approximation

Emergence of the continuum at $\gamma = \omega$

## "Slow-mode" approximation

$$H_{RS} = \omega a^\dagger a + \sigma_z(\Delta + \gamma a^\dagger a) + g\sigma_x(a^\dagger + a)$$

$$H_{RS}|\psi\rangle = E|\psi\rangle$$

$$|\psi\rangle = \psi_+(q)|+\rangle + \psi_-(q)|-\rangle$$

$$\sigma_x|\pm\rangle = \pm|\pm\rangle$$

$$a = \frac{1}{\sqrt{2}}\left(\zeta q + \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}q}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\zeta q - \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}q}\right), \quad \zeta = \sqrt{m\omega}$$

$$\left(\frac{m\omega^2}{2}\left(1 \pm \frac{\gamma}{\omega}\right) q^2 - \frac{1}{2m}\left(1 \pm \frac{\gamma}{\omega}\right)\frac{d^2}{dq^2} \pm \Delta'\right)\psi_\pm(q) + g\sqrt{2m\omega}q\,\psi_\mp(q) = E'\psi_\pm(q)$$

$$\Delta' = \Delta - \gamma/2, \quad E' = E + \omega/2$$

Emergence of the continuum at $\gamma = \omega$

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$$\Delta' = \Delta - \gamma/2, \quad E' = E + \omega/2$$

drop the kinetic term

Emergence of the continuum at $\gamma = \omega$

## "Slow-mode" approximation

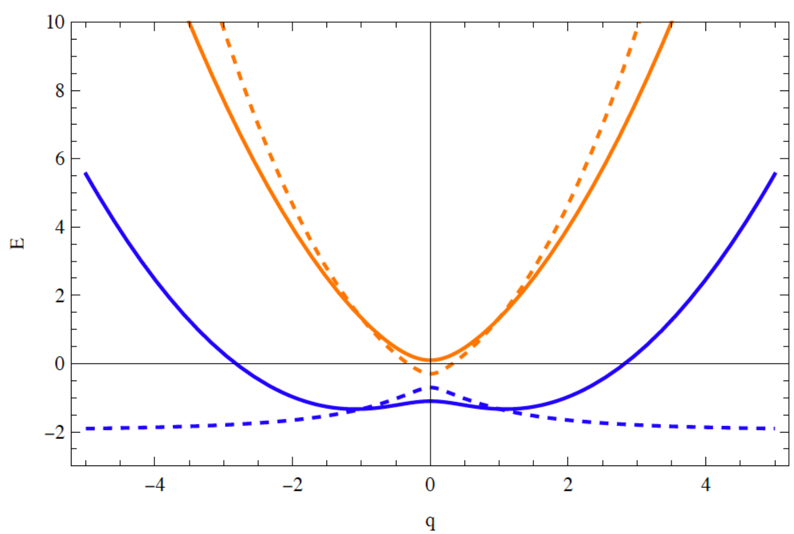
$$\left[\frac{m\omega^2}{2}\left(1 \pm \frac{\gamma}{\omega}\right) q^2 \pm \Delta' - E'\right] \psi_\pm(q) = -g\sqrt{2m\omega}q\,\psi_\mp(q)$$

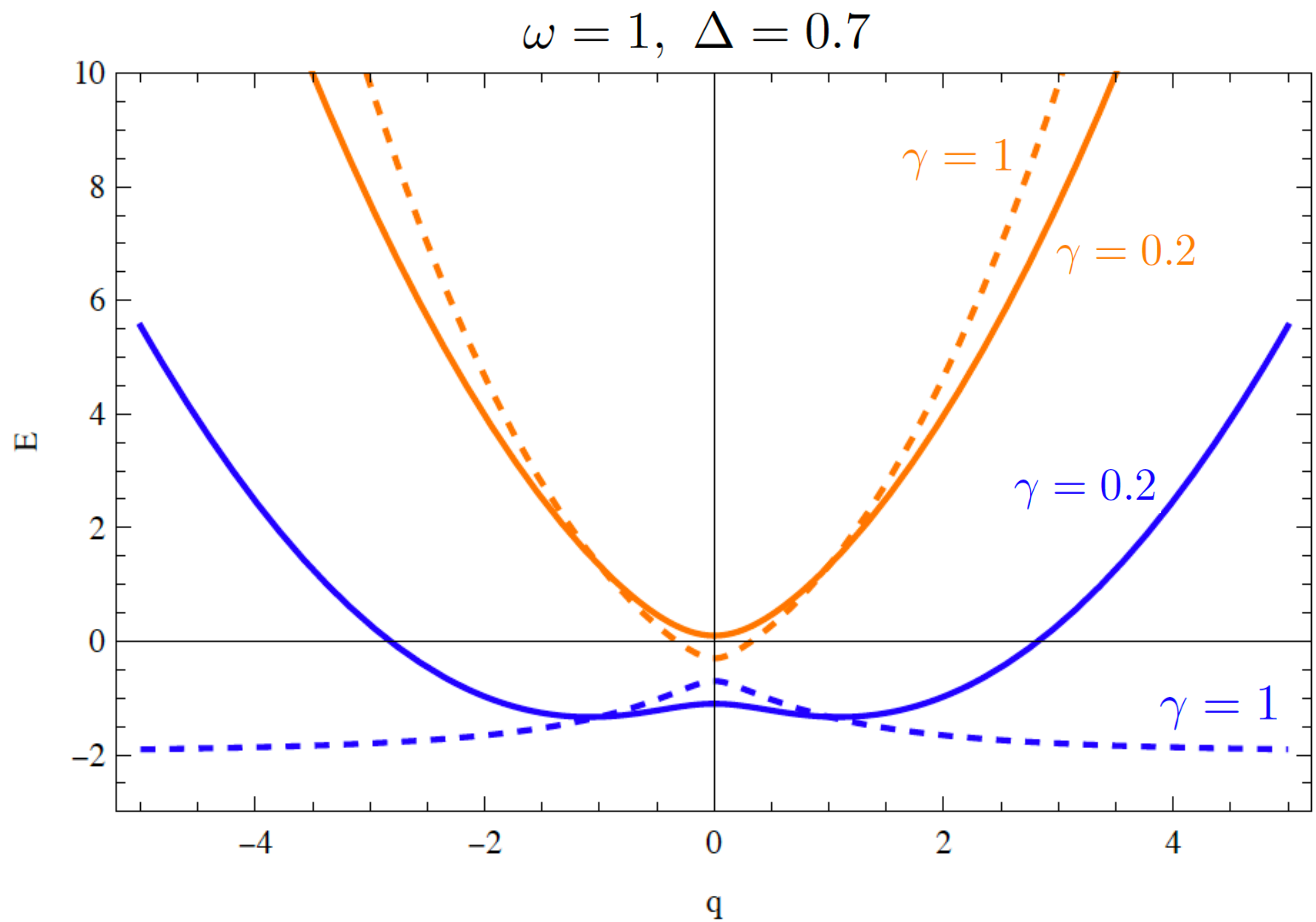
$$\Delta' = \Delta - \gamma/2, \quad E' = E + \omega/2$$

Eliminate $\psi_\pm(q)$ and solve for $E$ to obtain a pair of effective potentials

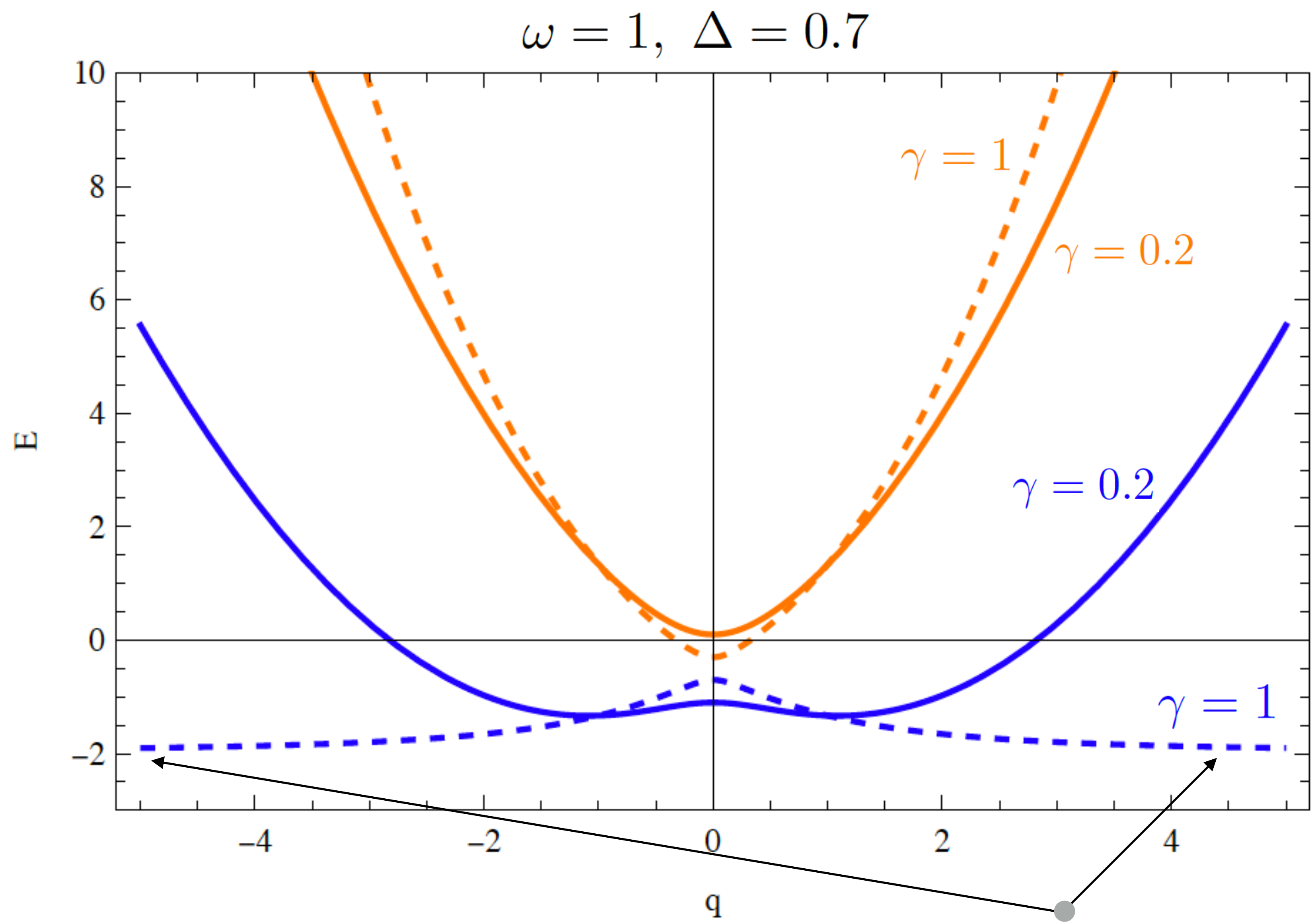
$$E_{\mathrm{upper}}(q) = \frac{m\omega^2}{2}q^2 - \frac{\omega}{2} + \left[\left(\Delta - \frac{\gamma}{2} + \frac{m\omega\gamma}{2}q^2\right)^2 + 2m\omega g^2 q^2\right]^{1/2}$$

$$E_{\mathrm{lower}}(q) = \frac{m\omega^2}{2}q^2 - \frac{\omega}{2} - \left[\left(\Delta - \frac{\gamma}{2} + \frac{m\omega\gamma}{2}q^2\right)^2 + 2m\omega g^2 q^2\right]^{1/2}$$





$\omega = 1, \ \Delta = 0.7$
10
8
6
4
2
0
-2
E
$\gamma = 1$
$\gamma = 0.2$
$\gamma = 0.2$
$\gamma = 1$
-4
-2
0
2
4
q

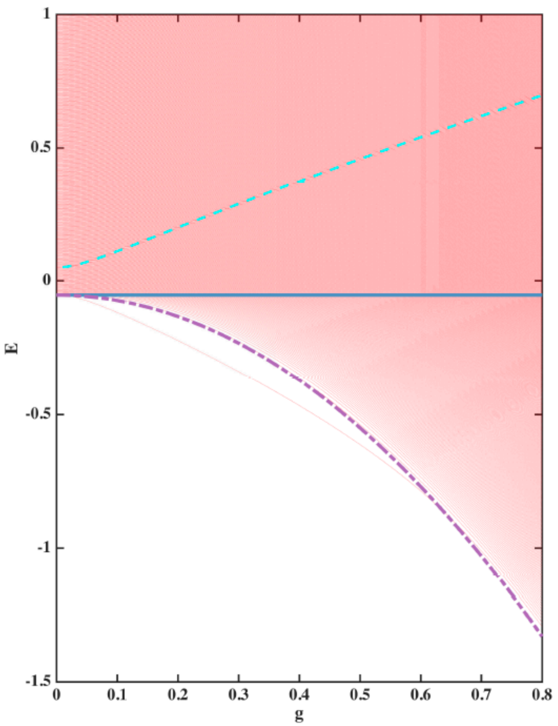



asymptotically flat effective potential

continuum of scattering states

# Summary/outlook

Analyses of the limit $\gamma \to \omega_-$ , numerics, and a ”slow-mode” approximation strongly suggest that the Rabi-Stark model supports a continuous spectrum with BICs.




Can the theory be regularized/renormalized to allow a full study of the spectrum *at* the point of instability?

How to *interpret* a continuous Rabi-Stark spectrum?

*Proof of concept* only, or physically relevant BICs?


D. Braak, L. Cong, H.-P. Eckle, HJ, E. K. Twyeffort, J. Opt. Soc. Am. B **41**, C97 (2024)
Feature issue: ”The Jaynes-Cummings model – 60 years and still counting”
arXiv:2403.16758