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Rollercoaster loop shapes

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Abstract

Many modern rollercoasters feature loops. Although textbook loops are often circular, real rollercoaster loops are not. In this paper, we look into the mathematical description of various possible loop shapes, as well as their riding properties. We also discuss how a study of loop shapes can be used in physics education.

Introduction

Have you ever looked closely at a rollercoaster loop (figure 1)? Have you noticed how the top may look like a half-circle, whereas the bottom looks different, with an increasing radius of curvature closer to the ground. Once you have noticed it, the reasons are probably obvious. We discuss first the riding properties of a rollercoaster including a circular loop, as a background to an analysis of other possible loop shapes. The Rollercoaster DataBase (RCDB) [1], includes many pictures of rollercoaster loops for comparison[1].

The circular vertical loop

The frictionless circular rollercoaster loop with negligible train length is a popular textbook problem. The speed is then obtained directly from the conservation of energy, i.e. $mv^2/2 = mg\Delta h$. At any given part of the frictionless rollercoaster, the centripetal acceleration is thus given by $a_c = v^2/r = 2gh/r$ where h is the distance from the highest point of the rollercoaster and r is the local radius of curvature.

Assume that you pass the top of a loop with a speed v_0 obtained, e.g., by starting from rest at a height $h_0 = v_0^2/2g$ above the top of the circular loop. If the loop has a radius r the centripetal acceleration at the top will be $a_0 = 2gh_0/r$. The centripetal accelerations at the side and at the bottom are immediately obtained from the value at the top as $a_0 + 2g$ and $a_0 + 4g$, respectively.

The limiting case of weightlessness ($0g$) at the top, where no force is needed between train and track, nor between riders and train, occurs when $h_0 = r/2$, so that the centripetal acceleration is given by $a_0 = g$ (and the centripetal force is thus provided exactly by the gravitational force from the Earth). The centripetal acceleration at the sides and bottom will be $3g$ and $5g$, respectively. (What is the *total* acceleration of the rider for these cases?) The corresponding 'g-forces' are $3g$ and $6g$.

Trains moving slowly across the top would fall off the track, were it not for the extra sets of wheels on the other side of the track. Similarly, the riders would depend on the restraints to remain in the train. However, even if the train were nearly at rest at the top, with riders hanging upside down, experiencing $-1g$, the riders would still be exposed to $5g$ at the bottom (and $2g$ at the side) if the loop were circular.

Children's rollercoasters may be limited to $2g$, family rides often reach $3g$, sometimes more, whereas many of today's large rollercoasters exceed $4g$. Depending on the individual's 'g-tolerance', the oxygen supply to the head may cease completely at 5–6g, resulting in

[1] The RCDB includes many more loop photos, illustrating different shapes. A few suggestions, in addition to those shown in figure 1, are Vortex at Paramount's Kings Island, Viper and Revolution at Six Flags Magic Mountain.

Figure 1. Examples of loop shapes. The red loop to the left is from Loopen at Tusenfryd in Norway (Vekoma, Corkscrew, 1988). The yellow loop in the middle is from HangOver (Vekoma, Invertigo, 1996) at Liseberg, now relocated to Sommerland Syd in southern Denmark. The loop to the right is from the newly opened 'Kanonen' (Launch Coaster, Intamin/Stengel, 2005) at Liseberg [1]. Exercise for the reader: The Kanonen train has a length of 9.5 m and takes about 1.3 s to pass the top of the loop. Use the photograph to estimate the g-force of the rider at the top of the loop. Does it make any difference whether you sit in the front, back or middle? How much?

unconsciousnes if extended in time. Although higher g-forces can be sustained with special anti-g-suits, e.g. for pilots, $6g$ for any extended period of time would not be acceptable for the general public [2].

However, the disadvantages of circular loops are not limited to the maximum g-force at the bottom. Entering the circular loop from a horizontal track would imply an instant onset of the maximum g-force (as would a direct transition to a circular path with smaller radius of curvature). An immediate transition from one radius of curvature to another would give a continuous, smooth track, but with discontinuous second derivatives. Clearly, a function with continuous higher derivatives would be preferable. From the loop photos in figure 1, it is obvious that different approaches have been used to achieve the desired transition from a smaller radius of curvature at the top to a larger radius at the bottom. Below, we discuss a number of possible loop shapes with this property.

Generation of alternative loop shapes

Curves of various shapes can be described through a set of differential equations, prescribing the derivatives of the position with respect to distance, s, along the curve.

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{r}.$$

This set of coupled differential equations can be used in a spreadsheet program or in an ordinary differential equation (ODE) solver, e.g. in Matlab. The difference between the different loop types is reflected in the expression for the curvature (i.e. $1/r$), as discussed below.

Loops with constant centripetal acceleration

One of the easiest alternative loop shapes to generate is one that gives a constant centripetal acceleration, as suggested, for example, in the classical engineering textbook by Meriam and Kraige [3]. If the centripetal acceleration $a_c = 2gh/r$ is to remain constant at a value $a_{c,0} = 2gh_0/r_0$ we find that

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{r} = \frac{1}{r_0}\frac{h_0}{h}.$$

Thus, the radius of curvature varies linearly with elevation. This condition could be applied for the whole loop or for just the lower part of the loop,

Figure 2. Different loop shapes for the condition of constant centripetal acceleration. The first two loop shapes give a centripetal acceleration of $2g$ and $3g$, respectively, throughout the loop (for a particular velocity), whereas the last two loops maintain these conditions only for the bottom part of the loop, matched to a 120° circular arc at the top. The loop with $3g$ centripetal acceleration throughout the loop is very similar to the Invertigo loop in figure 1.

which is then matched with circular track for the upper part, as illustrated in figure 2 for the cases of $a_c = 2g$ and $3g$, respectively. Figure 3 illustrates the influence of the matching angle on the loop shape for a constant centripetal acceleration $a_c = 3g$.

For this type of loop, the force from the track on the train increases continuously, as the component normal to the track of the gravitational acceleration, $g\cos\theta$, increases. The maximum g-force for the rider will thus be $a_c + g = (a_c/g + 1)g$.

The condition of constant centripetal acceleration results in a loop shape that is symmetric around the lowest point, and the loop shape could be extended and repeated to generate a sequence of loops. An example of a double loop can be found in the Great American Scream Machine from 1989 [1].

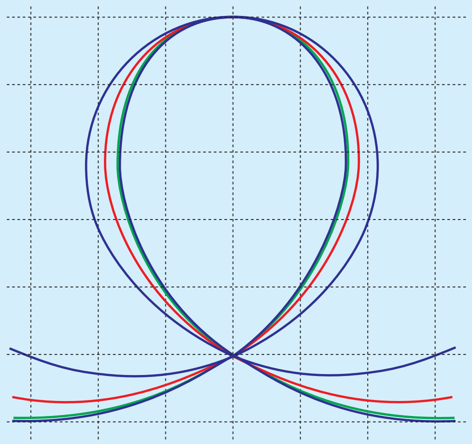

Figure 3. Loops generated using the condition of a constant centripetal acceleration $a_c = 3g$ matched to a circular arc covering the top 0°, 60°, 120° and 180°, respectively. The loops are normalized to give the same distance between the highest point and the 'intersection'. The widest loop is then obtained for the matching where the track is vertical and the narrowest when the condition is applied throughout the loop.

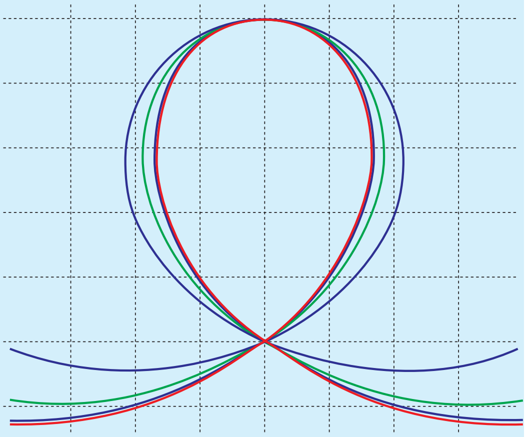

Figure 4. Loops generated using the condition of a constant g-force of $4g$ matched to a circular arc covering the top 0°, 60°, 120° and 180°, respectively. The loops are normalized to give the same distance between the highest point and the 'intersection'. Just as in the case of constant centripetal acceleration, the narrowest loop is obtained when the condition is applied throughout the loop.

Loops with constant g-force

Another possibility to avoid the sudden onset of large g-forces could be to design a loop with constant g-force, either throughout the loop or through part of it. Let the condition be applied below a point where the track forms an angle θ_0 with the horizontal, has a local radius of curvature r_0, and the centripetal acceleration is again given by $a_{c,0} = 2gh_0/r_0$.

The g-force factor is given by the force from the train on the rider divided by the weight of the rider, which can be expressed as a vector,

$$\boldsymbol{f} = m(\boldsymbol{a} - \boldsymbol{g})/mg = (\boldsymbol{a} - \boldsymbol{g})/g.$$

In the matching point to the circular arc the vector $\boldsymbol{a} - \boldsymbol{g}$ has the magnitude

$$\left(\cos\theta_0 + \frac{2h_0}{r_0}\right) g.$$

In order for the g-force to remain constant, the radius of curvature must depend on the height and slope of the track as

$$\frac{1}{r} = \frac{1}{h}\left(\frac{h_0}{r_0} + \frac{\cos\theta_0 - \cos\theta}{2}\right).$$

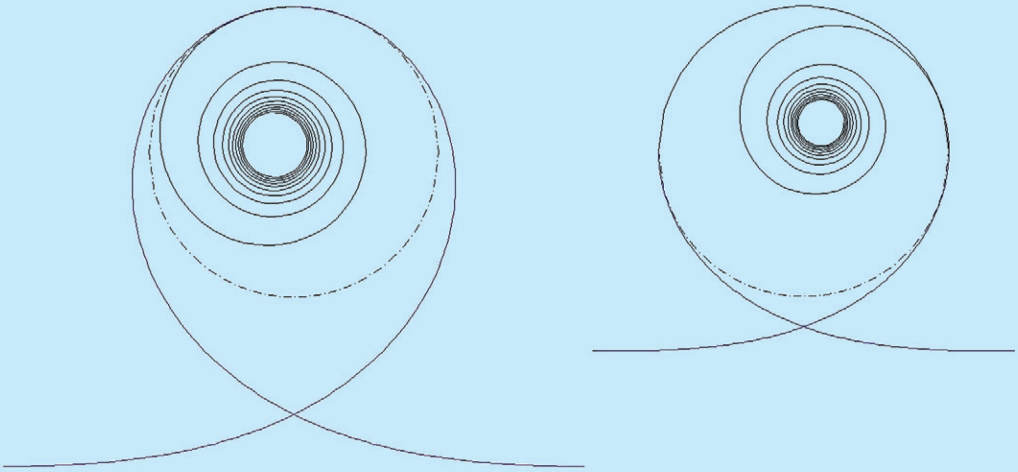

Figure 5. Examples of clothoid loops, where the track enters horizontally. In the loop to the left, clothoid extends throughout the loop, whereas in the loop to the right, the top is a half circle, matched to the Cornu spiral where the track is vertical. For these choices, the angle at the intersection is about 127° and 141°, respectively. The continuation of the spiral is shown as an illustration, together with the circle corresponding to the radius of curvature at the top.

This is then the condition to be inserted in the specification of the ordinary differential equation determining the curve. As in the case of constant centripetal acceleration, the curve is symmetric around the lowest point (figure 4) and the condition of constant g-force can be maintained throughout a sequence of loops—although the riding experience may not be that interesting.

Clothoids

The Cornu spiral, known from diffraction in a single slit, is the basis for clothoid loops, first introduced by Werner Stengel[2]. Parts of clothoids are also used to connect parts of tracks with different curvatures. Clothoids are frequently used in railway and road building, e.g. for motorway exits [4]. A driver keeping constant speed in a clothoid segment of a road can turn the steering wheel with constant angular velocity; the Cornu spiral has the property that the radius of curvature is inversely proportional to the distance, s, from the centre of the spiral. Although the property is easily expressed as $d\theta/ds = as$, some algebra is needed to express the boundary conditions in a convenient form, since the parameter, s, is not easily available in relation to the position in the loop. Integration of the angle gives $\theta = \theta_0 + as^2/2$. If the Cornu spiral is oriented with the centre sloping downwards the track will pass a lowest point between the loop and the centre of the clothoid. At this point the radius of curvature will be larger than at the top or the side of the loop. The ratios between these radii of curvature may be one way to specify the clothoid. Figure 5 shows examples of clothoid loops where the centre of the Cornu spiral is horizontal. The Looping Star was an early implementation of this shape, which also gives a good fit for the Kanonen loop.

[2] Werner Stengel was awarded an honorary doctorate at Göteborg University in 2005. More information about his work can be found at www.rcstengel.com and in a separate article in this issue. The history page of that website shows a model of the clothoid loop, which was used for the first time in the Revolution [1] looping coaster from 1976. The clothoid also gives a good fit, e.g. to the shape in the loop in Kanonen.

Discussion

So, how can an observer distinguish between different loop shapes? Their properties are not typical park advertising material, except possibly for maximum g-force for the rollercoaster as well as loop height (although a well defined reference level is not necessarily provided).

Can we find the critical parameters that specify a loop? The width at the widest point in the loop can provide a suitable length scale. The height difference between this ‘waistline’ of the loop and the highest point indicates if the upper part is a half-circle or whether an alternative loop shape is used also for this part. (A problem, however, is that it is often hard to find a photo angle that gives you a clean profile.) Other characteristic properties are the angle where the tracks 'intersect', the elevation of the top (and of the lowest point, if there is one) relative to the intersection. All these depend on the loop type and on the parameters chosen for each type.

Another key would be the riding properties. Accelerometer data [5] for Invertigo [1], for example, show that the g-forces at the bottom just before and after the loop are about $4.5g$ and $4.0g$, respectively, and about $2.2g$ at the top, consistent with an essentially constant centripetal acceleration of about $3.2g$ [3]. Comparison with the data for the reverse tour shows that the difference between the g-factors before and after the loop is due not to the shape of the track but to energy losses, which happen throughout the ride. The loop shapes discussed in this article reflect physics as the 'Art of systematic oversimplification'. The accelerometer data from the Invertigo ride immediately show that friction cannot be neglected.

To make a program to draw loops of various types, generate the corresponding loop and loop measures, and compare to a few real loops would constitute a challenging student project. A test of the fit can be obtained by printing a photo of the loop onto a transparency and then changing the size of the calculated graph on the computer screen (or resizing directly in a layer in a drawing program). Smaller computational projects can be to provide students with coordinates for a particular loop shape. If desired, coordinate may be revised to distinguish between 'heartline' and the track, both for the case of an ordinary coaster and a suspended ('inverted') one. Students can then be asked to work out, for example, radius of curvature as a function of elevation or angle, the time required to complete the loop and the time variation of the g-forces on the body[4]. For increased difficulty, energy losses may be taken into account.

[3] The values in the text, taken from [4], hold for the way back, just after the train has been pulled up to initial height and started its journey back. On the trip through the loop towards the end of the first half of the ride, the values are instead $4g$, $3g$ and about $0.5g$, demonstrating that the constant centripetal acceleration holds only for a particular speed at the top.

[4] Coordinates for various loops, or sample Matlab codes to generate them, can be provided to teachers.

A special aspect is the effect of the length of the train. In fact, the choice of loop shape is often related to the train length, as reflected in the loops in figure 1. (See also other loops in the RCDB [1].) For a longer train, the centre of mass lies further below the top, making a higher and narrower loop desirable [6]. Calculating loop shapes may permanently change the way you view a rollercoaster loop.

Acknowledgments

Duane Marden helped me find additional pictures of different loop shapes and brought to my attention the different looks of a clothoid loop introduced by Stengel and the more tear-drop loop shapes used, e.g., by Arrow. Clarence Bakken gave me access to electronic accelerometer data for Invertigo. Ulf Johansson at Liseberg and Werner Stengel provided me with glimpses of the complications involved in real rollercoaster loops. This work was partially funded by CSELT (Chalmers Strategic Effort in Learning and Teaching) and by RHU (The Swedish Council for Renewal of Higher Education).

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