Persistent currents through a quantum impurity: Protection through integrability

Johan Nilsson,[1] Hans-Peter Eckle,[2] and Henrik Johannesson[3]
[1]*Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
[2]*Advanced Materials Science, University of Ulm, D 89069 Ulm, Germany*
[3]*Department of Physics, Göteborg University, SE-412 96 Göteborg, Sweden*
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We consider an integrable model of a one-dimensional mesoscopic ring, with the conduction electrons coupled by a spin exchange to a magnetic impurity. A symmetry analysis based on a Bethe ansatz solution of the model reveals that the current is insensitive to the presence of the impurity. We argue that this is true for *any* integrable impurity-electron interaction, independent of the choice of physical parameters or couplings. We propose a simple physical picture of how the persistent current gets protected by integrability.

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The physics of quantum impurities has become an important chapter in the evolving theory of strongly correlated matter. The reasons are several: First, quantum impurity problems arguably represent the simplest settings in which to analyze various aspects of electron correlations. A case in point is the Kondo effect, where a magnetic impurity induces an effective electron-electron interaction that increases as the energy scale is lowered. Various extensions of the original problem have opened up entire new fields of investigations, from the modeling of correlated transport in DNA molecules, to scenarios for non-Fermi-liquid behaviors.[1] Secondly, quantum impurity problems are often tractable by exact analytical methods, most prominently the Bethe ansatz technique which exploits the integrability of the paradigmatic Kondo and Anderson models.[2] This has proven immensely useful, with the exact results serving as "benchmarks" for more conventional, numerical, or perturbative methods.

Most importantly, progress in experiments on mesoscopic and nanoscale systems now enables controlled studies of a single quantum impurity interacting with conduction electrons. In groundbreaking experiments in the late 1990s,[3] semiconductor quantum dots, connected capacitively to a gate and via tunnel junctions to electrodes, were shown to exhibit a tunable Kondo effect: Below a characteristic temperature T_K, a single electron occupying the highest spin-degenerate level of the dot forms a spin singlet with electrons in the leads, producing a Kondo resonance at the Fermi level. This and subsequent developments[4] have turned quantum impurity physics into an essential piece also of modern nanoscience.

An interesting problem in this context is how the charge persistent current (PC) in a mesoscopic ring coupled to a quantum dot is affected by a Kondo resonance. A PC is the equilibrium response to a magnetic Aharonov-Bohm flux piercing the ring.[5] It requires for its existence that an electron maintains its phase coherence while encircling the ring, and is thus expected to be sensitive to scattering off the quantum dot. In a previous study,[6] it was found that the PC is amenable to a Bethe ansatz analysis when the quantum dot is side-coupled to the ring. For certain privileged values of the flux, the problem was mapped onto the integrable one-dimensional (1D) Kondo model with a linearized dispersion. Contrary to expectation, it was found that the Kondo impurity that represents the dot has no effect on the persistent current. While this result conforms with those of some other authors,[7,8] a well-controlled renormalization group (RG) analysis[9] together with large-scale numerics[10] strongly suggest that the PC, in fact, vanishes when the ring is larger than the Kondo screening cloud (other related work includes Refs. 11 and 12). This raised doubts about the applicability of a Bethe ansatz approach.[13] Having served for many years as a work horse in the study of bulk quantum impurity physics, the 1D integrable Kondo model was now perceived to suffer from difficulty when applied to this particular problem: Its linear dispersion relation, in addition to decoupling spin and charge degrees of freedom, enforces a nonstandard procedure for extracting the PC from the finite-size spectrum. It was suggested that these features likely explain the failure to obtain an effect from the impurity on the PC.[13]

In an attempt to shed light on this intriguing issue, we investigate, in this Brief Report, the influence of a local magnetic moment on the PC in a mesoscopic ring, using an integrable model with a nonlinear dispersion relation for the electrons. As in Ref. 6, the impurity is coupled to the ring in such a way that the ring is unaffected when the coupling to the impurity is switched off. Unlike the analysis in Ref. 6, however, we do *not* linearize the electronic spectrum but keep the parabolic dispersion of nonrelativistic electrons since we want to expressly study its possible effect on the PC. Apart from the results implied by Refs. 9 and 10, the electronic band curvature is not known to play any significant role in the physics of the Kondo effect. Nevertheless, a small number of studies have addressed the question of nonlinearities in the electronic spectrum in the Kondo problem, with a particular eye on how to preserve the integrability of the model.[14–16] In what follows, we shall draw on some of the insights gained from these studies.

The basic building blocks that go into the construction of an integrable model are the two-particle scattering matrices S_{ij}.[17] These have to satisfy the Yang-Baxter equation

$$S_{ij}S_{ik}S_{jk} = S_{jk}S_{ik}S_{ij}, \tag{1}$$

the hallmark of integrability.[17] Constructing the electron-impurity scattering matrix by the same procedure as for the ordinary Kondo model[2] but now with a quadratic spectrum necessitates, for consistency, the introduction of a local potential term in the Hamiltonian:

$$V_c(x) \propto [\delta'(x^+) + \delta'(x^-)]x/|x|, \tag{2}$$

with $x=0$ being the location of the impurity. Moreover, to satisfy the Yang-Baxter equation (1) for electron-impurity and electron-electron scattering, the electrons must interact via a local interaction whose strength is adapted to the Kondo coupling of the magnetic moment. The inclusion of interacting electrons implies a dichotomy: attractive electron-electron interaction necessitates an antiferromagnetic Kondo coupling, while repulsive electron-electron interaction implies a ferromagnetic Kondo coupling. We shall concentrate here on the latter case. Since our interest is to study the consequences of a nonlinear band structure in the framework of an integrable model, both the auxiliary potential in Eq. (2) and the dichotomy between electron-electron interaction and Kondo coupling can be easily tolerated. We note in passing that a mechanism leading to a ferromagnetic Kondo coupling in quantum dots has recently been suggested by Silvestrov and Imry.[18]

The first-quantized Hamiltonian on a ring of circumference L, consistent with integrability as outlined above, is given by[16]

$$H = \sum_i [-\partial_{x_i}^2 + (J\vec{\sigma}_i \cdot \vec{\sigma}_0 + J')\delta(x_i)] + \sum_i V_c(x_i) + \sum_{i<j} 2c\,\delta(x_j - x_i), \tag{3}$$

where $2J=-c<0$ and $J'=-J$ are required by integrability. The integrability of the model allows for an exact solution, encoded by the Bethe ansatz equations (BAEs),

$$\frac{I_j}{L} = z_c(k_j) = \frac{k_j}{2\pi} - \frac{1}{2\pi L}\sum_{\gamma=1}^{N_s} \theta_{1/2}(k_j - \lambda_\gamma), \tag{4a}$$

$$\frac{J_\gamma}{L} = z_s(\lambda_\gamma) = -\frac{1}{2\pi L}\left[\sum_{j=1}^{N_c} \theta_{1/2}(\lambda_\gamma - k_j) - \sum_{\delta=1}^{N_s} \theta_1(\lambda_\gamma - \lambda_\delta) + \theta_{1/2}(\lambda_\gamma)\right]. \tag{4b}$$

Here, $k_j(\lambda_\gamma)$ are rapidities of the charge (spin) degrees of freedom, and $\theta_n(x) \equiv -2\tan^{-1}(x/nc)$. Note that, except for the last term on the right-hand side of Eq. (4b), these are the BAEs for the repulsive δ-function Fermi gas.[19] This last term, however, encapsulates the contribution of the localized magnetic moment. As is well known,[5] an Aharonov-Bohm flux threading the ring is equivalent to imposing twisted boundary conditions and, in our case, adds a term proportional to the flux to Eq. (4a). We shall include such a flux term into our analysis conveniently at a later stage. The quantum numbers I_j and J_γ are integers or half-integers depending on the number of electrons, $N_c=N_\uparrow+N_\downarrow$, and the number of down-spin electrons, $N_s=N_\downarrow$. Their maximal and minimal values are $I^\pm$ and $J^\pm$, respectively. Thus

$$N_c = I^+ - I^- + 1, \quad N_s = J^+ - J^- + 1, \tag{5a}$$

$$D_c = (I^+ + I^-)/2, \quad D_s = (J^+ + J^-)/2. \tag{5b}$$

N_c and N_s constitute the numbers of particles in the charge and spin Fermi seas. D_c and D_s are the numbers of electrons and down-spin electrons moved from the left to the right Fermi points of their respective Fermi seas.

It is important to note that in contrast to the standard 1D Kondo model with a linearized spectrum, charge and spin degrees of freedom are coupled through the BAE in Eq. (4). This suggests that the presence of the magnetic impurity may now feed back on the charge sector and affect the PC. To find out whether this happens requires a careful analysis, to be expounded in what follows.

Introducing the root density functions $\vec{\rho}=(\rho_c,\rho_s)^T$,

$$\rho_c(k) = \frac{\partial z_c(k)}{\partial k}, \quad \rho_s(\lambda) = \frac{\partial z_s(\lambda)}{\partial \lambda}, \tag{6}$$

in the charge (c) and spin (s) sectors, we can employ the well-known framework[20] for extracting the lowest order finite-size corrections to the ground-state energy in the thermodynamic limit. These are the finite-size corrections which determine the PC. Using the Euler-Maclaurin formula for converting sums into integrals, one can retain finite-size corrections (in principle, to arbitrary order) when converting the BAE (4) into a set of inhomogeneous coupled linear integral equations for the root densities.[20] The phase shifts of Eqs. (4) translate, via Eq. (6) and the Euler-Maclaurin formula, into the integral kernel and the inhomogeneity of these integral equations, respectively. Given that the phase shifts are odd functions of their respective arguments, the inhomogeneities and, therefore, the solutions of the integral equations also attain a certain definite symmetry. Our analysis of the PC will eventually rely exclusively on exploiting this symmetry. To achieve a finite-size energy expression correct to order $1/L$, we introduce the integration limits $(k^\pm,\lambda^\pm)$ by

$$Lz_c(k^\pm) = I^\pm \pm 1/2, \quad Lz_s(\lambda^\pm) = J^\pm \pm 1/2, \tag{7}$$

such that Eqs. (5) become

$$\frac{N_c}{L} = \int_{k^-}^{k^+} dk\rho_c(k), \quad \frac{N_s}{L} = \int_{\lambda^-}^{\lambda^+} d\lambda\rho_s(\lambda), \tag{8a}$$

$$\frac{D_c}{L} = z_c(0) + \frac{1}{2}\left[\int_0^{k^-} dk + \int_0^{k^+} dk\right]\rho_c(k), \tag{8b}$$

$$\frac{D_s}{L} = z_s(0) + \frac{1}{2}\left[\int_0^{\lambda^-} d\lambda + \int_0^{\lambda^+} d\lambda\right]\rho_s(\lambda), \tag{8c}$$

where, from Eq. (4),

$$z_c(0) = \frac{1}{2\pi}\int_{\lambda^-}^{\lambda^+} \rho_s(\lambda)\theta_{1/2}(\lambda)d\lambda, \tag{9a}$$

$$z_s(0) = \frac{1}{2\pi}\left[\int_{k^-}^{k^+} dk\rho_c(k)\theta_{1/2}(k) - \int_{\lambda^-}^{\lambda^+} d\lambda\rho_s(\lambda)\theta_1(\lambda)\right]. \tag{9b}$$

The formal solution of the integral equations can then be decomposed as

$$\vec{\rho}(k,\lambda) = \vec{\rho}_\infty(k,\lambda) + \frac{1}{L}\vec{\rho}_d(k,\lambda) + \frac{1}{24L^2}\left[\frac{\vec{\rho}_1(k,\lambda|k^\pm,\lambda^\pm)}{\rho_c(k^+)} + \frac{\vec{\rho}_1(-k,-\lambda|-k^\mp,-\lambda^\mp)}{\rho_c(k^-)} + \frac{\vec{\rho}_2(k,\lambda|k^\pm,\lambda^\pm)}{\rho_s(\lambda^+)} + \frac{\vec{\rho}_2(-k,-\lambda|-k^\mp,-\lambda^\mp)}{\rho_s(\lambda^-)}\right]. \tag{10}$$

Here, $k^\pm$ and $\lambda^\pm$ play the role of Fermi points of the charge and spin excitations. The densities in Eq. (10), therefore, depend on the numbers $I^\pm$ and $J^\pm$, or, via Eq. (5), on the parameters N_r and D_r $(r=c,s)$. The form of Eq. (10) depends crucially on the fact that the charge rapidities k_j enter with odd symmetry into the BAE (4). The term $\vec{\rho}_d/L$ describes the finite-size contribution of the magnetic moment, with $\vec{\rho}_\infty$, $\vec{\rho}_d$, $\vec{\rho}_1$, and $\vec{\rho}_2$ solving the integral equations with appropriate inhomogeneous parts. In particular,

$$\vec{\rho}_{\infty 0} = \begin{bmatrix} 1/2\pi \\ 0 \end{bmatrix}, \quad \vec{\rho}_{d0} = \begin{bmatrix} 0 \\ K_{1/2}(\lambda) \end{bmatrix}, \tag{11}$$

with the kernel $K_{1/2}(\lambda) = d\theta_{1/2}(\lambda)/d\lambda$.

From these solutions, we obtain the ground-state energy to first order in $1/L$ as[20]

$$E_0 = L\epsilon_\infty + \epsilon_{d\infty} + \frac{1}{L}\left\{v_c\left[\frac{\Delta N_c{}^2}{4\xi^2} + \xi^2\Delta_D^2 - \frac{1}{12}\right] + v_s\left[\frac{(\Delta N_c - 2\Delta N_s)^2}{4} + \frac{1}{2}(\Delta D_s)^2 - \frac{1}{12}\right]\right\}, \tag{12}$$

where $\Delta_D \equiv \Delta D_c + \Delta D_s$ is the sum of the finite-size deviations of D_c and D_s from their bulk ground-state values. Equation (12) is obtained by an expansion of the ground-state energy around the thermodynamic limit $L\to\infty$, which, in our formalism, is tantamount to an expansion in terms of $(k^\pm \pm k_0)$ and $(\lambda^\pm \pm \lambda_0)$ around symmetric integration limits k_0 and λ_0 [cf. Eq. (7)]. This is followed by a transformation of variables from $k^\pm$ and $\lambda^\pm$ to X_r $(r=c,s$ and $X=\Delta N,\Delta D)$ evaluated at $k^\pm = \pm k_0$ and $\lambda^\pm = \pm\lambda_0$, thereby incurring as the Jacobian matrix of the transformation a "dressed charge" matrix,[20] which can be shown to obey the same integral equations with the unit matrix as inhomogeneity.[20] In our case, ξ is the function that parametrizes this matrix. v_c and v_s, finally, are the Fermi velocities in the charge and spin sectors. Note that the leading contribution to the ground-state energy due to the magnetic moment is given by $\epsilon_{d\infty}$. This contribution has the character of a boundary term.[21] However, the magnetic moment also affects the parameters in the $1/L$ term, as becomes obvious from the decomposition (10) when inserted into Eqs. (8) and (9).

Next, we analyze the finite-size energy to obtain an expression for the equilibrium response of the system to an externally applied magnetic flux, i.e., the persistent current in the presence of the local magnetic moment. In fact, the persistent current is precisely determined by the finite-size contributions proportional to $1/L$ in the energy (12), and is obtained by taking the derivative of E_0 with respect to the external Aharonov-Bohm flux. Trading the flux ϕ for twisted boundary conditions via a gauge transformation leads to an additional shift in the number D_c of electrons moved from the left to the right Fermi points in the charge Fermi sea: $\Delta D_c \to \Delta D_c + \phi$. Using this replacement in Eq. (12), we arrive at the formal result for the persistent current:

$$I(\phi) = -\frac{e\xi^2 v_c}{\pi L}[\Delta D_c + \Delta D_s + \phi]. \tag{13}$$

Now we are in a position to answer the central question of this investigation: How does the presence of the magnetic moment, which interacts with the electrons in the ring, influence the persistent current? Equation (13) tells us that, to answer this question, we need to analyze the effect of the magnetic moment on the parameters ΔD_c and ΔD_s. According to Eq. (8), the parts of ΔD_c and ΔD_s stemming from the magnetic moment are given by (we ignore bulk terms)

$$\Delta D_c^d = z_c^d(0) + \frac{1}{2}\left[\int_0^{k_0} dk + \int_0^{-k_0} dk\right]\rho_c^d(k), \tag{14a}$$

$$\Delta D_s^d = z_s^d(0) + \frac{1}{2}\left[\int_0^{\lambda_0} d\lambda + \int_0^{-\lambda_0} d\lambda\right]\rho_s^d(\lambda), \tag{14b}$$

where the density functions $\rho_c^d(k)$ and $\rho_s^d(\lambda)$ are solutions of the integral equations with the inhomogeneity $\vec{\rho}_0^d$ [cf. Eq. (11)], and we have symmetric integration limits $k^\pm = \pm k_0$ and $\lambda^\pm = \pm\lambda_0$ in the ground state according to our discussion after Eq. (12).

There is, however, no need to explicitly solve the integral equations to obtain further insight into the quantities ΔD_c^d and ΔD_s^d. They follow simply from considering the symmetry of the functions involved. The symmetry properties of all functions derived from the integral equations follow from the basic odd symmetry of the Bethe ansatz charge rapidities k and the symmetry of the inhomogeneity. The inhomogeneity $\vec{\rho}_{d0}$ [cf. Eq. (11)] is even, as are all integral kernels. Further scrutiny, therefore, reveals that the symmetries are such that ΔD_c^d and ΔD_s^d both vanish. For example, $\vec{\rho}_d$ is an even function in both variables k and λ, and hence, from Eq. (9), $z_c^d(0) = z_s^d(0) = 0$ such that, moreover, from Eq. (14), $\Delta D_c^d = 0$ and $\Delta D_s^d = 0$. Hence there is no influence of the magnetic moment on the persistent current.

We reiterate that our result follows immediately from a symmetry analysis of the rapidities and the integral kernels implied by the BAE (4). Since these are generic, the result carries over to any model of a quantum impurity coupled to electrons with a dispersion relation that is an even function of momentum, i.e., for example, a parabolic (nonrelativistic) band, or, if defined on a lattice, a tight-binding band. The reason for this universal behavior of integrable quantum impurities is that the details of the model do not affect the generic symmetry of the Bethe ansatz, as demonstrated by

our analysis above. This is also consistent with results obtained for the supersymmetric *t-J* model, where the finite-size ($\propto 1/L$) contribution to the energy due to twisted boundary conditions was found to be independent of the presence of an integrable impurity.[22]

What is the physics behind this remarkable phenomenon? We propose that an answer may be constructed as follows: As is well known, integrable quantum dynamics in one dimension supports only forward scattering.[17] It is also known that a forward scattering phase shift of a free electron wave function incurred from a local static potential has no effect on a persistent current: As was shown by Gogolin and Prokof'ev,[23] there is a subtle cancellation [to $\mathcal{O}(1/L)$] of contributions to the persistent current from phase shifted states, leading to an expression for the current in terms of the Fermi level transition amplitude only. Provided that the effect of a quantum impurity on the conduction electrons can be faithfully encoded by a potential scatterer (much as in Nozière's local Fermi-liquid theory of the ordinary Kondo effect[24])—*and* that this property is not corrupted on a mesoscopic scale—our result would get an elegant and transparent explanation.

From an experimental point of view, one may be concerned that the protection of the persistent current is not robust against any deviation from integrability. That is, any small perturbation would make a difference, regardless of the perturbation being relevant or irrelevant in the sense of renormalization group theory. Thus, to detect a pure protected current will require "fine tuning" of the experimental setup so as to make sure that the dynamics remains integrable.

In conclusion, we have demonstrated on quite general grounds that there is no influence from a quantum impurity on the persistent current in a mesoscopic ring when the electron-impurity interaction is integrable. We conjecture that this result can be traced back to a cancellation of phase shifted contributions to the persistent current, in analogy to the simple case of noninteracting electrons in the presence of a single forward scattering local potential. To put this conjecture on a firm ground, and to extract implications for other Aharonov-Bohm (or Aharonov-Casher[6]) geometries, is an interesting and challenging problem.

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