Topological quantum numbers in nonrelativistic physics

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Abstract

Voltage measurements using the ac Josephson effect and electrical resistance measurements using the quantum Hall effect are capable of very high precision, despite the relatively poor control of details of the devices. Such measurements rely on topological quantum numbers, which, unlike symmetry-based quantum numbers, are insensitive to deviations of the system from ideality. The circulation in superfluid ^{4}He, flux quantization in superconductors and quantized Hall conductance are all examples of topological quantum numbers, but only the last two are known to be very precise. Vinen's early measurement of quantized circulation was based on measurement of the resulting Magnus force, and we (Ping Ao, Qian Niu and I) have recently shown that the strength of the Magnus force can itself be determined by an argument that shares common features with topological arguments.

I. SYMMETRY-BASED AND TOPOLOGICAL QUANTUM NUMBERS.

The quantum numbers with which we are most familiar are related to the algebra of symmetry operations. From Noether's theorem we know that a symmetry in classical dynamics leads to a conserved quantity, and for a compact symmetry group this translates in quantum mechanics to sets of discrete eigenvalues for the operator. The relations between rotational symmetry, conservation of angular momentum, and the algebraic structure of the angular momentum operators are things that most of us have taught in various classes. We also know that where the symmetry is broken, as it is when a magnetic field is applied to the atom or when the atom is embedded in a solid, the quantum numbers are immediately mixed with one another, and may no longer be identifiable with certainty. Other examples of such symmetry-based quantum numbers are isospin, parity, and the labels which occur in the nonrelativistic theory of hydrogen-like atoms, based on representations of the group $O(4)$. In all these cases a broken symmetry leads to mixing of the quantum numbers.

Topological quantum numbers are very different. There is strong experimental evidence that the quantum numbers can be determined with very high precision, and that they are insensitive to imperfections of the apparatus used to measure them. The Josephson voltage-frequency relation [1,2], which depends on the precise quantization of magnetic flux in a superconducting ring, is so precise that it has superseded and corrected earlier secondary standards of voltage, yet on a microscopic scale a Josephson junction appears as an irregular insulating barrier between two superconductors. Despite this, two differently constructed junctions made from different materials are found to be consistent to parts in 10^{17} [3,4]. Similarly, a quantum Hall bar provides a resistance standard far more reproducible than earlier standard resistances, and differently constructed quantum Hall devices agree to within a few parts in 10^{10} [5], but quantum Hall devices have geometrical parameters that depend on gates laid down with imperfect control at some distance from the interface where the two-dimensional electron layer occurs.

II. SUPERFLUIDS AND SUPERCONDUCTORS.

The argument for quantized circulation in a neutral superfluid was given in a rather cryptic form by Onsager [6]. A superfluid such as superfluid ^{4}He is characterized by a condensate wave function

$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})| \exp[iS(\mathbf{r})] \tag{1}$$

which is a single valued function of position $\mathbf{r}$, a one-particle wave function shared by all, or at least a finite proportion, of the particles in the system. The superfluid velocity is given by

$$\mathbf{v}_s = \hbar \mathbf{grad} S/m \ , \tag{2}$$

where m is the atomic mass. The phase S need not be single valued, but it can change by a multiple of 2π on a closed path that goes round either some obstacle, such as a solid wire running across the container, or when it goes round some mathematical line singularity on which the order parameter vanishes. The circulation of the superfluid velocity round such a path is given by

$$\oint \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m} \oint \mathbf{grad} S \cdot d\mathbf{r} = n \frac{h}{m} \ , \tag{3}$$

where n is the winding number of the phase of the order parameter around the obstacle or line singularity. The superfluid circulation is therefore quantized in units of h/m. This winding number is the simplest example of a topological quantum number.

Vinen [7] measured the circulation of superfluid round a wire by running a thin stretched wire down the axis of a cylindrical container of liquid helium, which was set into rotation before being cooled from the normal state to the superfluid state. The Magnus force on the wire split the nominally degenerate modes of vibration of the wire into two (ideally) circularly polarized components, and the frequency shifts measured the strength of the Magnus force, and so gave the magnitude of the circulation round the wire.

One might think that the quantum number n for the circulation round the wire simply corresponds to the angular momentum per helium atom, and that would be the case for an ideal system, but in fact the wire was very far from having a uniform circular cross-section. Not only were the normal modes not in fact degenerate, but scanning electron microscopy of the wires revealed a surface that looked more like a particularly rugged piece of mountain terrain than a smooth cylinder. The fact that quantization was observed illustrates the insensitivity of such topological quantum numbers to details of the geometry. The precision of the quantization shown by the experiment was only about 3%, and later experiments have only improved on this by about one order of magnitude [8].

For a superconductor, a superfluid in which the mobile particles are charged, the current density is given by

$$\mathbf{j} = -(\frac{e\hbar}{m}\mathbf{grad}S + \frac{2e^2}{m}\mathbf{A})|\Psi|^2 \ , \tag{4}$$

where a factor of $2e$ has been put in front of the vector potential $\mathbf{A}$ to allow for the fact that Ψ represents the condensate wave function for pairs of electrons. The Meissner effect ensures that the current density falls off exponentially into the interior of the superconductor. In this interior region where the current density vanishes, eq. (4) gives

$$\oint \mathbf{A} \cdot d\mathbf{r} = -\frac{\hbar}{2e} \oint \mathbf{grad}S \cdot d\mathbf{r} = n\frac{h}{2e} \ . \tag{5}$$

Since the integral of the vector potential round a ring gives the flux enclosed by the ring, this relation shows that the flux trapped by a superconductor is equal to n times $h/2e$, where again n is the winding number of the phase of the condensate wave function. The path enclosing quantized flux has to be in a region free of current density, and it may either surround regions in which there is no superconducting material, where the flux is concentrated, or, for a type II superconductor in a weak magnetic field, it may surround flux lines where the superconducting order parameter has singularities.

Flux quantization was originally predicted by London [9], but without the factor of 2 in the denominator. The correct form given by the BCS theory was given by Byers and Yang [10], and by Brenig [11]. Although the earliest measurements of flux quantization were not precise, subsequent developments with the Josephson effect have shown that flux is indeed quantized with very high precision, under circumstances in which the geometry is very irregular.

The high precision of flux quantization in superconductors contrasts strongly with the relatively poor precision with which the quantization of circulation in neutral superfluids has been checked. This is not just because it is easier to detect a small change in magnetic flux than it is to compare circulation in different superfluid containers, although that is true. For the superconductors the relation between the topological invariant, the winding number, and the physical quantity, the trapped flux, is subject to corrections which are exponentially small, such as the penetration of the magnetic field into the superconductor which is governed by the London equation. For the neutral superfluid the connection between the topological invariant and the measured quantity has corrections which are powers of physical dimensions, rather than exponentials, because the governing equations have the structure of the Laplace equation instead of the London equation.

III. QUANTUM HALL EFFECT.

The discovery of the quantum Hall effect was particularly dramatic because not only was it unpredicted by theory, but the first publication showed that the ratio of current to voltage was an integer with a precision of the order of 1 part in 10^5 [12]. The fact that the quantized Hall conductance is a topological quantum number was implicit in Laughlin's explanation of the effect [13], and was made explicit in later papers [14–17].

Laughlin's argument can be regarded as a generalization of Bloch's famous theorem that for a loop of conductor threaded by a magnetic flux Φ the free energy is a periodic function of Φ, and that the current $\partial F/\partial\Phi$ is periodic with period h/e and has zero average. For the quantum Hall effect the loop is replaced by an annular region of two-dimensional conductor, with a solenoid passing through the hole in the middle to carry some flux which can be changed without changing the magnetic field in the conducting region. If the magnetic field has such a strength that the Fermi energy lies in a gap between the energies of the mobile states that characterize the Landau levels it is possible to maintain different electrochemical potentials μ_i and μ_o on the two edges of the annulus, since the longitudinal conductance is close to zero, and very little current flows between the two edges. One imagines these electrochemical potentials maintained by contact with two reservoirs of electrons. Under these circumstances, change of the flux Φ by h/e restores the ring to its original quasiequilibrium state, except that there are trivial gauge changes of the electron wave functions by the factor

$$\exp[-i(e/\hbar)\int \delta\mathbf{A}(\mathbf{r})\cdot d\mathbf{r}]\ . \tag{6}$$

However, there may also be n electrons transfered from one reservoir to the other, changing the free energy of the system by $n(\mu_o - \mu_i) = ne\,\delta V$, where δV is the voltage difference between the two edges. At the same time, the change in flux generates a voltage $d\Phi/dt$ round the annulus, and this does work $\int J\, d\Phi = \bar{J}h/e$, where $\bar{J}$ is the average current around the annulus. The equality of the work done and the change in free energy then gives

$$ne\,\delta V = \bar{J}h/e\ , \tag{7}$$

so that the ratio of current to voltage is quantized in multiples of e^2/h.

For the fractional quantum Hall effect Gefen and I have argued that there is a broken symmetry, so that the change of flux by one quantum unit h/e does not restore the original quasiequilibrium state, but that a change qh/e must be made in the flux to restore the original state [18], where q is the denominator of the fraction. From this point of view the quantized Hall conductance can be regarded as the ratio of two winding numbers, the denominator related to the change in the winding number of the electron wave functions round the annulus, and the numerator related to the number of electrons that are cranked from one edge to the other as a consequence. As with the quantization of flux in a superconductor, one expects to be able to make corrections to the quantized value exponentially small, because corrections come from such processes as direct tunneling of electrons from one edge to the other, excitation of free quasiparticles to modify the current, and deviations of the current J from its average value $\bar{J}$. These should be exponentially small if the width of the annulus is sufficiently large, the temperature is sufficiently low, the Fermi energy is sufficiently far from the Landau levels, and the length of the edges is sufficiently large.

In fact, of course, precise measurements of the quantum Hall effect are not made in this geometry of an annulus. The alternative approach due to Avron and Seiler [15] and to Niu *et al.* [16], in which the Hall conductance is represented as a Chern number,

$$\sigma_{xy} = \frac{e^2}{2\pi h}\int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta (\langle \frac{\partial\psi}{\partial\alpha} \bigg| \frac{\partial\psi}{\partial\beta}\rangle - \langle \frac{\partial\psi}{\partial\beta} \bigg| \frac{\partial\psi}{\partial\alpha}\rangle) , \tag{8}$$

can be translated more readily into the situation that exists in a Hall bar. Here α, β are gauge parameters defined across the current leads and voltage leads respectively, and ψ is the wave function of the Hall system.

IV. MAGNUS FORCE IN NEUTRAL SUPERFLUIDS.

Vinen's experiment to measure the quantized circulation in superfluid ^{4}He [7] relies on a generalization of the classical Magnus formula for the transverse force on a vortex, or an object around which the fluid circulates, moving relative to the fluid. The same method has recently been used to measure quantized circulation in the B phase of superfluid ^{3}He [19]. A standard way of deriving this Magnus force in classical hydrodynamics is to consider a fixed vortex, around which the circulation is κ, with fluid flowing past it with velocity $\mathbf{v}_F$, and then consider the momentum balance on a cylinder of very large radius whose axis coincides with the vortex. The forces on this cylinder are the force holding the vortex in position and the force due to the Bernoulli pressure on the outside of the cylinder, which is perpendicular to the axis of the cylinder and to $\mathbf{v}_F$, and has magnitude $\rho\kappa v_F/2$ per unit length. The momentum density in the cylinder is constant under these steady state conditions, so the net force on the cylinder must equal the momentum flux out of the cylinder, which again has magnitude $\rho\kappa v_F/2$ per unit length, but is oppositely directed to the force due to the Bernoulli pressure. The total force per unit length is therefore

$$\mathbf{F}_M = \rho\kappa\hat{\mathbf{n}} \times (\mathbf{v}_V - \mathbf{v}_F) , \tag{9}$$

where $\kappa = \oint \mathbf{v}\cdot d\mathbf{r}$ is the circulation of the fluid round the vortex, whose direction is given by the unit vector $\hat{\mathbf{n}}$, and I have used the Galilean invariance of the system to include the possibility of the vortex itself moving with velocity $\mathbf{v}_V$.

In the case of a superfluid there are some questions about how this formula generalizes to the two-fluid dynamics which is believed to describe superfluid ^{4}He, and more that arise when systems of fermions are considered. The simplest generalization says that only the superfluid component participates in the rotational motion around the vortex, so the fluid density ρ should be replaced by the superfluid density ρ_s, while the fluid velocity $\mathbf{v}_F$ should be replaced by the superfluid velocity $\mathbf{v}_s$. There are at least four questions that have been raised about this way of doing things:

1. Should the total fluid density ρ be used or the superfluid density ρ_s?

2. Iordanskii [20] showed that phonons (or rotons) flowing past a vortex are scattered asymmetrically, so that they produce a transverse force which depends on the normal fluid density ρ_n. Is this an extra force that must be included, and how big is it?

3. Kopnin and Salomaa [21] and Volovik [22] have argued that in a fermion superfluid there are gapless states in the vortex core, and these will lead to a term proportional to $\kappa \times (\mathbf{v}_V - \mathbf{v}_n)$ that may almost completely cancel the usual Magnus force.

4. Can these ideas be generalized to a charged superfluid (superconductor), where the London screening leads to disappearance of the circulating current at large distances?

The form of the transverse force quoted by Donnelly [23] is

$$\mathbf{F}_t = \kappa \hat{\mathbf{n}} \times [\rho_s(\mathbf{v}_V - \mathbf{v}_s) + (\rho_n - \sigma)(\mathbf{v}_V - \mathbf{v}_n)] \, , \tag{10}$$

The term proportional to ρ_n gives the Iordanskii force, and the term in σ comes from phonon scattering.

The Magnus force is a nondissipative force that is linear in the vortex velocity, so it corresponds to a term in the Lagrangian that is linear both in velocity and position. Such a term gives rise to a term in the action that is path-dependent, but is independent of the speed at which the path is traversed. In quantum mechanics this gives rise to a path-dependent phase, and this is just what is known as a Berry phase [24], as was observed by Haldane and Wu [25]. An interesting aspect of this, which we developed some years ago [26], is that at zero temperature, for a process in which the vortex moves from some initial path and eventually returns to it, this leads to a Berry phase equal to 2π times the number of atoms that the surface traced out by the vortex encloses. If eq. (10) is correct, this is reduced by a factor ρ_s/ρ at nonzero temperatures, and this is a quantity whose significance as a quantum number is less obvious.

We have recently given a first principles derivation of the expression for the Magnus force on a quantized vortex [27]. For the sake of simplicity I describe here a two-dimensional superfluid, but the generalization to three dimensions is straightforward. We consider a single isolated vortex in an otherwise uniform homogeneous superfluid film. This is a state with broken symmetry, since the vortex can be anywhere in the system, but a pinning force can be introduced by adding a potential $\sum_i V(\mathbf{r}_i - \mathbf{r}_0)$. Any repulsive potential will serve to localize the vortex, since the particle density is lower at the vortex core, but the pinning potential should be strong enough to prevent escape of the vortex by thermal fluctuations or by quantum tunneling. The position $\mathbf{r}_0$ of the pinning center can be a function of time. In our paper we calculate the average force exerted by the particles on the pinning center when the pinning center is moving with constant velocity $\mathbf{v}_V$, using the instantaneous eigenstates of the system with fixed pinning potential as a basis. It is perhaps more satisfactory to give the pinning center an exponential acceleration, by taking its coordinates to be

$$x_0 = \frac{v_V}{\gamma}(e^{\gamma t} - 1), \quad y_0 = 0 \; . \tag{11}$$

To lowest order in the vortex velocity this gives

$$\mathbf{F} = \sum_\alpha f_\alpha \left\langle \Psi_\alpha \left| \nabla_0 H \frac{i\hbar \mathcal{P}_\alpha}{E_\alpha - H - i\gamma} \mathbf{v}_V \cdot \nabla_0 + \text{h.c.} \right| \Psi_\alpha \right\rangle \; ; \tag{12}$$

here the Ψ_α are the instantaneous eigenstates with energies E_α and occupation probabilities f_α, $\mathcal{P}_\alpha$ is the projection operator off the eigenstate α, and ∇_0 is the gradient with respect to the position of the pinning center. Since $\nabla_0 H$ is the commutator of the operator ∇_0 with H, the commutator cancels the energy denominator, in the limit $\gamma \to 0$, and eq. (12) can be rewritten as

$$\mathbf{F} = -i\hbar \hat{\mathbf{n}} \times \mathbf{v}_V \sum_\alpha f_\alpha (\langle \frac{\partial \Psi_\alpha}{\partial x_0} \left| \frac{\partial \Psi_\alpha}{\partial y_0} \rangle - \langle \frac{\partial \Psi_\alpha}{\partial y_0} \right| \frac{\partial \Psi_\alpha}{\partial x_0} \rangle) \; , \tag{13}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the plane. This expression is the familiar form of the integrand that defines a Berry phase [24].

Because of the translation invariance of the system, the operator ∇_0 is equivalent to $-\sum_i \nabla_i$, and eq. (13) can therefore be written in terms of the total momentum operators $\mathbf{P}$ of the particles in the system, in the form

$$\mathbf{F} = -\frac{i}{\hbar}\hat{\mathbf{n}} \times \mathbf{v}_V \sum_\alpha f_\alpha \langle \Psi_\alpha \left| [P_x, P_y] \right| \Psi_\alpha \rangle \ . \tag{14}$$

The expectation value of the commutator that appears here can be written as the curl of the momentum density, and so Stokes' theorem can be used to write the result as

$$\mathbf{F} = \frac{i\hbar}{2}\hat{\mathbf{n}} \times \mathbf{v}_V \oint d\mathbf{r} \cdot [(\nabla - \nabla')\rho(\mathbf{r}', \mathbf{r})]_{r=r'} \ , \tag{15}$$

where the integral is taken over any path which goes round the vortex at a sufficiently large distance from it. In the equilibrium state of the superfluid with a single stationary vortex only the superfluid component contributes to this integral, since the interaction between the phonons (or rotons) leads to a normal fluid viscosity which prevents the normal fluid from sharing the circulation of the superfluid. This integral therefore leads to the form

$$\mathbf{F} = \rho_s \kappa \hat{\mathbf{n}} \times \mathbf{v}_V \ , \tag{16}$$

which only agrees with the form quoted in eq. (10), under the boundary conditions $\mathbf{v}_s = \mathbf{v}_n = 0$ which we have used, if the parameter σ is equal to ρ_n. This form seems to fit the data on the effect of temperature in the vibrating wire experiments [7,28], but there is some uncertainty in the interpretation of this data in the presence of the strong damping that occurs at higher temperatures.

A different argument by Wexler [29] has shown that the coefficient of $-v_s$ in the transverse force, is also ρ_s, which, from Galilean invariance, leaves no room for a transverse force proportional to the normal fluid velocity v_n.

These results answer the first and third question which I listed at the beginning of this section. The coefficient of $\mathbf{v}_V \times \kappa\hat{\mathbf{n}}$ in the transverse force is the superfluid density, not the total fluid density. Our derivation would appear to be valid for a fermion superfluid of the BCS type, so there does not seem to be any special contribution from the vortex core states to this force. Indeed, experimental measurement of the Magnus force exerted by the B phase of superfluid ^{3}He shows the usual form at very low temperatures [19]. Stone [30] has analysed the arguments for such a contribution, and does not find the effect in the absence of an explicit mechanism for transferring momentum from the core to the background [21,22].

Although the Magnus force does not appear as a topological quantity in this argument, the expectation value of the commutator of two components of momentum is related to the kind of quantity that appears in the topological expression for the Hall conductance.

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