Quantum Matter: Concepts and Models

homework problems, course week 6, spring 2020

Please structure your solutions carefully. All essential steps in your analysis and calculations should be made explicit.

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1. The one–dimensional Hubbard model

The Hubbard model in one dimension describes N_e electrons on a lattice of N lattice sites which are allowed to hop to nearest–neighbour sites. Double occupancy of a site with two electrons (which, of course, must have opposite spin orientation) costs an energy U. The second quantized Hamiltonian for this model is:

$$\begin{aligned}\mathcal{H} = & - t \sum_{\substack{j=1,\\ \sigma=\pm1}}^{N} \left\{ \psi_\sigma^\dagger(x_j)\psi_\sigma(x_j+a) + \psi_\sigma^\dagger(x_j)\psi_\sigma(x_j-a) \right\} \\ & + \frac{U}{2} \sum_{\substack{j=1,\\ \sigma=\pm1}}^{N} \left\{ \psi_\sigma^\dagger(x_j)\psi_\sigma(x_j)\psi_{-\sigma}^\dagger(x_j)\psi_{-\sigma}(x_j) \right\}, \end{aligned} \tag{1}$$

where $x_j = ja, j = 1, 2, \dots N$ and a is the lattice spacing. The Fermi operators obey anti–commutation relations

$$\left\{ \psi_\sigma(x_j), \psi_{\sigma'}^\dagger(x_l) \right\} = \delta_{\sigma,\sigma'}\delta_{jl}.$$

a) Diagonalize the hopping part of this Hamiltonian by Fourier transformation,

$$\psi_\sigma(x) = \frac{1}{\sqrt{N}} \sum_k \exp(ikx) c_{k,\sigma},$$

and determine the allowed k–values assuming periodic boundary conditions $x_{j+N} = x_j$.

b) Show that

$$\mathcal{M}_\uparrow = \sum_j \psi_\uparrow^\dagger(x_j)\psi_\uparrow(x_j)$$

and $\mathcal{M}_\downarrow$ defined analogously are conserved quantities, i.e. commute with $\mathcal{H}$. Therefore $\mathcal{M}_\uparrow$ and $\mathcal{M}_\downarrow$ provide quantum numbers $M_\uparrow$ and $M_\downarrow$ of the spectrum of $\mathcal{H}$,

$$E = E(M_\uparrow, M_\downarrow; t, U).$$

c) Determine the symmetries of $E = E(M_\uparrow, M_\downarrow; t, U)$ under the transformations

$$\psi_\sigma(x_j) = (-1)^j c_{j,\sigma}$$

and

$$\psi_\uparrow(x_j) = (-1)^j c_{j,\uparrow}^\dagger, \quad \psi_\downarrow(x_j) = c_{j,\downarrow}.$$

Are the new operators in both cases again Fermi operators?

2. Tensor products of R-matrices

a) Warm-up: Show that for two spins we have

$$\begin{aligned}\sigma_1 \cdot \sigma_2 &= \sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z \\ &= \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

b) Using the convention (10.12) in Hans-Peter's book for partitioning an R–matrix into 2×2 blocks (see link on the course homepage to *lecture notes,* Feb 20), show that the vertex weight of two vertices, (10.19), can be written as a tensor product of R–matrices in block form. Writing the elements of this composite object, we have

$$\sum_{\gamma_2=\pm} R_{\alpha_1}^{\alpha_1'}(\gamma_1, \gamma_2) R_{\alpha_2}^{\alpha_2'}(\gamma_2, \gamma_1') = (R_1 \otimes R_2)_{\alpha_1 \alpha_2}^{\alpha_1' \alpha_2'}(\gamma_1, \gamma_1')$$

where

$$R_n = \left(R_{\alpha_n}^{\alpha_n'}(\gamma_n, \gamma_n') \right) = \begin{pmatrix} R_+^+(\gamma_n, \gamma_n') & R_+^-(\gamma_n, \gamma_n') \\ R_-^+(\gamma_n, \gamma_n') & R_-^-(\gamma_n, \gamma_n') \end{pmatrix}$$

are the R–matrices of the two vertices $n = 1, 2$.

Note that the vertex weights w_{ij} entering the R–matrices R_1 and R_2 are the same for both R–matrices.