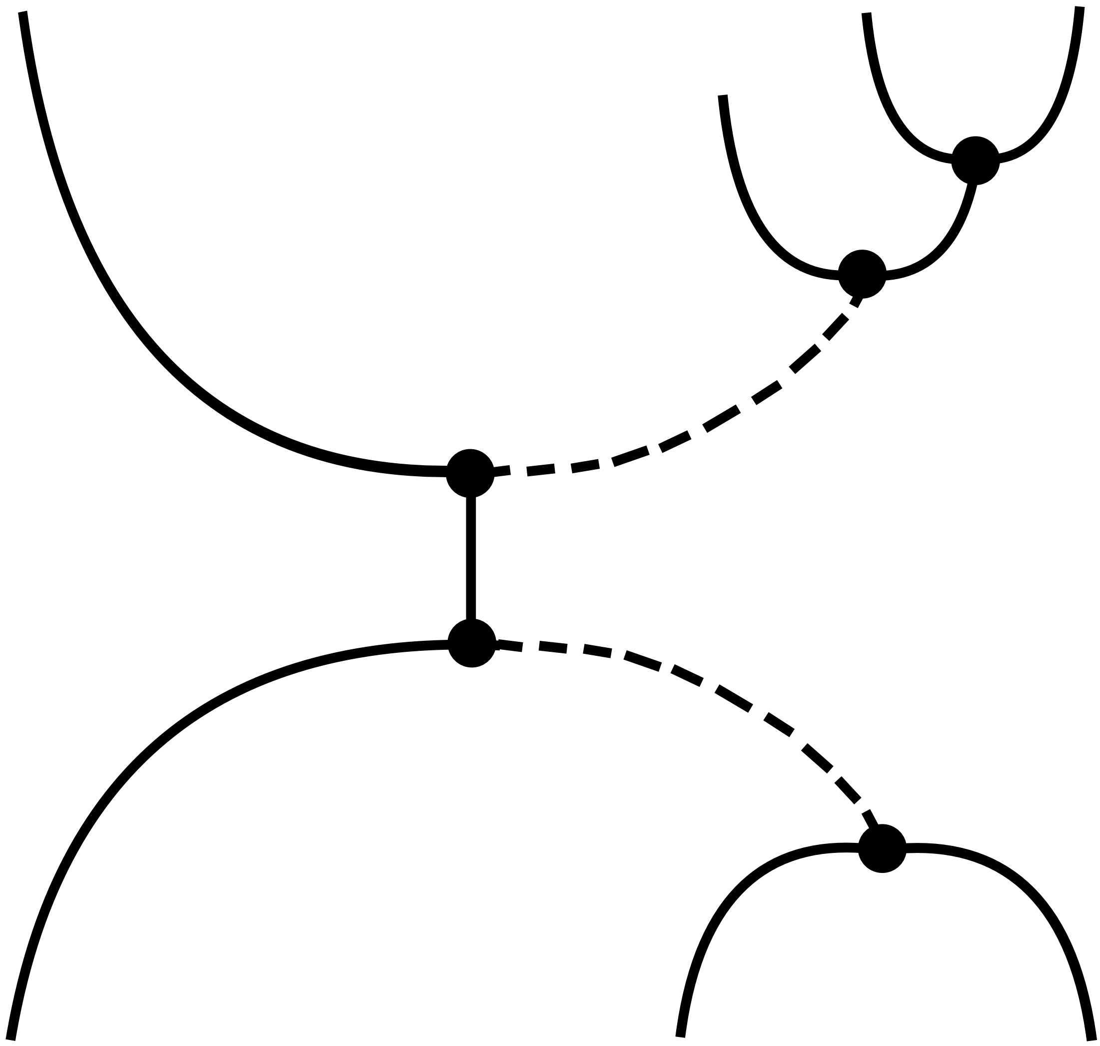
UNIVERSITY OF GOTHENBURG



Construction of Physical Models from Category Theory

Master Thesis in Theoretical Physics

Marko Marjanovic

Department of Physics
University of Gothenburg
Gothenburg, Sweden 2017

Construction of Physical Models from Category Theory

Marko Marjanovic

Supervisor:
Henrik Johannesson

Examiner:
Mats Granath

May 19, 2017
Department of Physics
Gothenburg University

Author: Marko Marjanovic
Email:
gusmarjma@student.gu.se

Department of Physics
Gothenburg University
SE-412 96 Göteborg
Sweden
+46 (31) 772 1000

Abstract

This thesis is a literature study which aims to present an introduction to category theory in a way suitable for the practicing physicist. We particularly focus on monoidal categories from which we construct physical models in a novel and abstract fashion. Special consideration is given to the monoidal category **Set**, which has sets as objects, functions as morphisms and the tensor product as its monoidal structure, and the monoidal category **FdHilb**, which has finite dimensional Hilbert spaces as objects, linear maps as morphisms and the tensor product as its monoidal structure. We then propose a physical interpretation regarding these categories, namely, **Set** as being a model for classical physics and **FdHilb** as being a model for quantum physics. Using the mathematics of category theory to structurally compare these categories guides us towards the categorical proof of the no-cloning theorem in quantum information theory. We also provide a detailed discussion regarding the monoidal category **2Cob**, which has manifolds as objects, cobordisms as morphisms and the disjoint union as its monoidal structure. The monoidal categories **2Cob** and **FdHilb** resemble each other far more than either resembles **Set**. Shared features are particular kinds of internal comonoids, compact structure and therefore also similar diagrammatic calculus. Together these features leads us towards topological quantum field theories. We also discuss concrete categories, real-world categories, abstract categories, why quaternionic quantum mechanics is excluded within the framework of category theory and how group-representations in fact are category-theoretic constructs.

Acknowledgements

First and foremost I would like to thank my thesis advisor Prof. Henrik Johannesson. The door to Prof. Johannesson's office was always open whenever I had a question about my research or writing. He consistently allowed this paper to be my own work, but steered me in the right direction whenever he thought I needed it. He also, patently, proofread my work several times and gave me both valuable advice and encouragement throughout my time as his student for which I am truly grateful.

I would also like to thank my college and friend Sebastian Östnell for his very valuable comments on this thesis. I have always enjoyed our stimulating discussions regarding both physics and other topics, and I feel gratefully indebted to him for his continuous support and friendship throughout my years of study.

Finally, I must express my very profound gratitude to my parents for their unfailing support and encouragement throughout my whole life, to my two brothers for continuously reminding me of what the most important things in life are, and last but not least to my grandmother and late grandfather whom I always thought of as an extra pair of parents. This accomplishment would not have been possible without them. Thank you.

Marko Marjanovic

Contents

1

Introduction

A new mathematical foundation with the aim of unifying mathematical constructs within a single framework has in recent years emerged. Category theory was formulated in 1945 by Saunders Mac Lane and Samuel Eilenberg with the goal of understanding certain kind of maps which preserves structures within algebraic topology [1]. Following the success it was soon realised that this mathematical approach could expand as to unify many known mathematical constructs within several mathematical areas such as algebra, geometry, topology, analysis and combinatorics! Indeed, category theory is a very general theory and its premise is that mathematical constructs can be organized into distinct types of categories. These categories all include a family of objects and between any two given objects a set of maps that are restricted by composition rules which will be established in the next chapter.

What makes category theory so powerful for uniting mathematical constructs is that it categorises mathematical constructs in a manner such that theorems and proofs can be generalised for all mathematical constructs within the same categorisation. The crucial thing that allows for this type of reasoning is that maps within category theory are as important as the objects themselves. This is in contrast to the more conventional way of understanding mathematical constructs, namely set theory. Given any set, the most fundamental questions are: ”how many elements does the set have and how are they related?”. However, between any two objects in any category the most fundamental questions are rather: ”which objects are we considering and which maps exists between them?”.

By imposing certain structures on categories, special types of categories known as *monoidal categories* can be constructed. Monoidal categories are, from the physicists point of view, the most important aspects of category theory and are thus given a comprehensive treatment in this thesis. The reason why monoidal categories are such great tools stems from the fact that physical systems cannot be studied in isolation, that is, we can only observe systems via other systems, such as a measurement apparatus. What monoidal categories let us do is to group these individual systems into compound systems such that models regarding the systems in question can be constructed by virtue of the mathematics of category theory. The objects and maps in monoidal categories can furthermore be visualised graphically since any monoidal category has graphical calculus incorporated within it which is a very useful feature regarding monoidal categories [2]. This allows for an intuition which is very powerful and can be used to prove theorems in a rather trivial way compared to algebraic methods.

Monoidal categories are currently used in research regarding *topological quantum computing* [3]. Let us take a short digression in order to discuss the main idea. Most of what is explained can be found in greater detail in the excellent review paper by Nayak et al. from 2008 [4].

In the research field regarding quantum computing, physicists make use of two-level quantum systems called qubits. Ordinary bits in classical computers are also two-leveled but the crucial thing that differentiates between bits and qubits is that qubits can be in a superposition of states and make use of entanglement. These features, restricted to qubits, can be used in order to make some calculations far less time consuming than they are for classical computers. However, a fundamental issue regarding qubits is that entanglement is very sensitive to disturbances, thus making these more efficient calculations hard to manage within a time-scale necessary to actually compute anything of interest. Research regarding topological quantum computing propose a solution to this problem. The suggestion is to represent qubits by so-called 'topological non-trivial configurations' instead. The idea is that the topology will keep the configuration from falling apart, meaning that they become much more stable towards external disturbances. In experimental set-ups, topological non-trivial configurations are achieved by manipulating certain kinds of particles called *anyons* [5]. These particles can only exist in two-dimensional systems and have special topological properties which can be deduced from the operation of exchanging the position of two of them which is referred to as *braiding*. The braiding of anyons is mathematically modelled through the so-called 'braid group' $\mathcal{B}_n$ where n stands for the number of particles within the two-dimensional system. Given a system of anyons, the braid group is used to specify the braiding of them so that a 'fusion rule' can be constructed. The fusion rule is crucial since it is the result of 'fusing' anyons together which generates the topological non-trivial configurations which can be represented as qubits in this approach to quantum computation. If the number of anyons would have been constant, then $\mathcal{B}_n$ would be sufficient in order to specify the topological properties of the anyons. In practise, however, since we create, destroy and fuse anyons, the number of them continually changes which means that a single representation of $\mathcal{B}_n$ is not enough. What we need is representations of $\mathcal{B}_n$ for all values of n which are compatible with each other and with fusion which is exactly what a certain type of monoidal category known as a 'modular tensor category' provides!

As an example, in order to make this more concrete, let us examine a two-dimensional system where only one type of anyon exist, namely Fibonacci anyons (the simplest anyon which can constitute a 'universal quantum computer'). This example can be found in Blass and Gurevich's paper from 2015 [6], where they moreover explain the use of the modular tensor category in greater detail. The fusion rule is given by

$$\tau \otimes \tau = 1 \oplus \tau.$$

This equation represent the result of fusing together two Fibonacci anyons. Either we get 1 (vacuum) or τ (a Fibonacci anyon). This fusion rule can alternatively be incorporated as part of the structure of some particular modular tensor category. From the fusion rule, one might think that we are dealing with a two-dimensional Hilbert space in which the general result is a superposition of these two states. However, within the modular model, the superposition is rather the *ways* in which one can obtain a single Fibonacci anyon following the fusion rule. Indeed, if we start with three Fibonacci anyons

$$\tau \otimes \tau \otimes \tau,$$

we can fuse the first two to either get τ or vacuum. If the result is vacuum, then the overall result is one τ, namely the one we did not fuse. However, if we instead obtain τ, then this Fibonacci anyon can be fused with the third one which might produce yet another τ. So we indeed end up with two possible ways of producing one τ according to whether the first two Fibonacci anyons fused to vacuum or τ. Another possibility for obtaining "two possible ways" is by instead fusing the last two Fibonacci anyons and then the result with the first. Now, according to the modular model, the fusion of the three Fibonacci anyons generates a two-dimensional Hilbert space of "ways to obtain a Fibonacci anyon". Moreover, this Hilbert space will differ in basis depending on which two Fibonacci anyons we started fusing. Within our modular tensor category, the relation between these two ways of fusing is also part of its structure which is one of the reasons why this modular tensor category is a useful tool for studying Fibonacci anyons. Another reason for its usefulness is that braiding also is part of the structure. Indeed, yet another possibility for fusing three Fibonacci anyons would be to begin by fusing the first and third Fibonacci anyons. The modular tensor category representation of this possibility would use its braiding structure to move the first anyon to be adjacent to the third (or vice versa), and it would depend on the path along which that anyon is moved around the second one. In this regard, the modular tensor category can be thought of as a set of mathematically represented axioms for which the processes involving Fibonacci anyons obey. That is, it tells you the algebraic structure that one needs in order to formulate an anyon model. However, this novel approach to quantum computing has only recently started and only time will tell if it proves to be fruitful. We will give a rigorous definition of the notion of a monoidal category in chapter 2.

Another potential use of monoidal categories is within the theory of 'networks' (wires connected to boxes of some sort) [7]. Networks occurs within many sciences including biology, chemistry, engineering, control theory, and physics. They are used in order to better understand certain processes like for example particle interactions in physics, where Feynman diagrams are used to prescribe rules that govern the interactions, or chemical reactions in chemistry where so-called 'Petri nets' are used. Since networks occur in so many different scientific areas, the hope is that the graphical calculus incorporated in category theory could act as a general language which underlies them. The idea is then that established results within some network in a specific scientific area can be transferred to other similar networks within other sciences using the general language of graphical calculus. Category theorists have recently realised that networks with specified inputs and outputs can be seen as maps in some monoidal category [7]. These are networks that describe open systems, that is, systems in which for example energy, matter, signals or information can flow in and out and where systems can combine to form larger systems by composition and 'tensoring', that is, larger systems can be built by composing systems sequentially or in parallel. In order to generalise these networks, some particular network (process) will be maps in some monoidal category. The relation between the inputs and outputs can then be established by so called 'monoidal functors' (see section 5.5) which are special kinds of maps between monoidal categories. Functors in general are of great importance within category theory and will be given a rigorous treatment in this thesis.

Perhaps where monoidal categories have had the most applied impact is in the fields of logic and computer science. However, these fields make use of some concepts of category theory that will not be explained in this thesis for example the concept of monads which are of great interest in computer science for constructing functional programming. Essentially, within the context of computer science, a monad is used as a way of composing simple components in a way so that a more complex computer program can be implemented [8]. For any readers interested in these topics we refer to [9], [10].

In this thesis we will rather focus on the implementation of category theory to quantum theory. Quantum theory is established upon the mathematics of Hilbert spaces. A Hilbert space is a complex vector space together with a 'inner product' (a map between vector spaces). Within the framework of category theory, both complex vector spaces and the inner product can be incorporated into a monoidal category and therefore it is possible to use monoidal categories to probe quantum theory. A case in point, which we will elaborate upon in chapter 4, is the no-cloning theorem in quantum information theory. Furthermore, monoidal categories can also be used to investigate the "behaviour" of the tensor product $\otimes$ which is of the utmost importance in quantum theory. Indeed, what separates quantum theory from classical physics is that $\otimes$ allows for joint systems which cannot be decomposed to states of the individual subsystems. Within the framework of category theory, the tensor product occurs as a so-called *monoidal product* (see section 2.4.2) in the category **FdHilb**, the category which concerns finite-dimensional Hilbert spaces. A particularly interesting result, which will be discussed in chapter 4, is that the tensor product behaves very differently compared to the corresponding monoidal product in the category **Set**, which is the category which have sets as objects and functions as maps. This is interesting since **Set** is the category which contains the information upon Hilbert spaces are built!

As it turns out, the monoidal product in **FdHilb**, which has to do with quantum theory, behaves much like the monoidal product in the category **nCob**, which has to do with general relativity. This observation has led to the proposition that category theory may have a role in the goal of unifying quantum mechanics with general relativity [11]. The work relies on so-called 'topological quantum field theories' (TQFTs), which possess certain features one expects from a theory of quantum gravity [12], [13]. However, TQFTs are in this regard used as "baby models" in which calculations merely gives insight into a more full-fledged theory of quantum gravity, so there is still a lot of work to be done. Nevertheless, the mathematics concerning monoidal categories are used in this approach to essentially represent TQFTs as "translation maps", which 'translates' the categorical structure defined over **nCob** and then maps accordingly to the categorical structure defined over **FdHilb** [14]. Indeed, TQFTs (from the mathematicians point of view) is defined through a set of axioms [15] which category theorists have been able to express as the categorical notion of a 'monoidal functor'. The definition of a TQFT as a monoidal functor will be discussed in greater detail in chapter 5 of this thesis.

However, we will start this thesis by first stating the axioms of category theory, developing the theory such that familiar mathematical concepts can be cast entirely in categorical language before we use this information to

- construct categorical models,
- probe quantum theory, and,
- define TQFTs within the framework of category theory.

1.1 Purpose and Outline

The purpose of this project is to introduce category theory in a way suitable for the practising physicist and to show that the study of monoidal categories can give insights into physics in a unique and powerful way. This will be demonstrated by rigorous definitions and proofs together with examples and comments about the category in question. We will apply category theory to quantum mechanics and develop the theory to the point where we can specify how *topological quantum field theories* relates to category theory and along the way demonstrate the richness of the subject.

Much of this relatively recent field of study is usually covered fragmentary or at the research level. We feel there is a need for a self-contained text introducing categorical quantum mechanics in a more pedagogical manner which we hope this project provides. We also believe that category theory will take on a larger role within the physics community in the future and hope that this thesis will convince the reader why.

In chapter 2, we will state the axioms of category theory and focus on different ways one may think about categories. This will include categories as *concrete categories*, *real-world categories* and *abstract categorical structures*. We will also show how category theorists thinks about the notion of group as an *abstract categorical structure* and state some of the most important categories which will be used in this thesis, including **Set**; which has all sets as objects and functions between sets as morphisms, $\mathbf{FdVect}_{\mathbb{K}}$; which has all finite dimensional vector spaces over the field $\mathbb{K}$ as objects and linear maps between them as morphisms, **FdHilb**; which has all finite-dimensional Hilbert spaces over the complex field as objects and linear maps between them as morphisms, and the *strict monoidal category*; which is used to axiomatize real-world categories. The definition of a *functor*, which is a very important concept in category theory, will also be stated together with some examples.

In chapter 3 we will introduce *non-strict monoidal categories* so that we can axiomatize mathematically modelled processes which occur in real-world categories. Differences between strict monoidal categories and *non-strict monoidal categories* will be explained which will lead to the concept of *natural isomorphism*. We will also introduce the notion of *graphical calculus* for monoidal categories which allow for complicated proofs to be intuitive. Graphical calculus also gives a intuitive way of dealing with mathematically modelled processes of real-world categories. In addition we also show how Dirac notation can be fully incorporated into category theory in a natural way and why quaternionic quantum mechanics is prohibited within the framework of category theory.

The main result of chapter 4 is a proof of the no-cloning theorem of quantum infotmation theory. In order to come to this result we begin this chapter by showing how we can add additional *tensor structure* to monoidal categories in order to specify mathematically modelled processes used for some particular real-world systems. This will lead to the concept of *compact categories* which will make it possible to define the *monoidal categories* **Rel**, which have all sets as objects and all relations between them as morphisms, and **2Cob**, which have all two-dimensional manifolds as objects and all cobordisms between them as morphisms. This will lead us to the the notion of *Cartesian categories* which will make it possible to define the Cartesian category **Set**. With both compact categories and Cartesian categories established, we will be able to investigate the monoidal product in greater detail. Some monoidal products act differently than others and we will explain how this work. Moreover, Cartesian categories leads to the concept of *universal property* which is of great importance since it is a property which characterises an object uniquely.

In chapter 5 we will introduce the concepts of *internal comonoids* and *internal monoids* which allow us to investigate the *inner-structure* of categories. This will allow us to establish the main result of this chapter, namely how the standard definition of a TQFT can be expressed as a monoidal functor. However, in order to come to this conclusion we will first investigate the category **Cat**, which have all categories as objects and functors between categories as morphisms. This will lead to the very important notion of *natural transformation* which together with a few additional definitions regarding functors will lead us to our final result, how to express TQFT as a monoidal functor.

2

The Categorical Worldview

In the founding paper of category theory [1], Eilenberg and Mac Lane wrote that their main motivation for introducing the concept of a category was to be able to introduce the concept of functors which in turn where created to introduce the concept of natural transformations. Natural transformations will be developed in chapter 5 while the notions of categories and functors will be introduced in this chapter.

This chapter serves mainly as an overview. We will develop concepts which are necessary in order to apply category theory to quantum theory in a consistent way. In particular, we need to create a framework in which notions we are interested in are formalized purely within the language of category theory [16], [17], [18], [19], [20], [21], [22], [23].

2.1 Axioms of Category Theory

We will use the following syntax to denote a function:

$$f : X \longrightarrow Y :: x \longmapsto y$$

where X is a set of arguments which gets mapped to another set Y of possible values via the process of mapping specific arguments $x \in X$ to specific values $f(x) = y \in Y$.

Definition (Category). A category $\mathbf{C}$ is defined as to contain the following information:

1. A family[1] $|\mathbf{C}|$ of objects,

2. For any objects $A,B \in |\mathbf{C}|$, a set $\mathbf{C}\,(A,B)$ of *morphisms*[2] which is referred to as the *hom-set*,

[1] By family we typically mean class rather than set since a set containing all sets as objects is ill-defined due to 'Russell's paradox' [18]. However, the distinction will not be of any substantial importance in this project.

[2] The word 'morphism' traces back to the original paper on category theory [1]. Since category theory is a mathematical foundation built on first principles, effort has been made to use original nomenclature. By 'morphism' one means 'some sort of unspecified relation' between objects in a category.

3. For any objects A,B,$C \in |\mathbf{C}|$, and any $f \in \mathbf{C}(A,B)$ and $g \in \mathbf{C}(B,C)$, a *composite*[3] $g \circ f \in \mathbf{C}(A,C)$, that is, there exists a *composition operation* such that:

$$- \circ - : \mathbf{C}(A,B) \times \mathbf{C}(B,C) \longrightarrow \mathbf{C}(A,C) :: (f,g) \longmapsto g \circ f$$

   which is *associative* and include *unit morphisms*, that is,

   i. for any $f \in \mathbf{C}(A,B)$, $g \in \mathbf{C}(B,C)$ and $h \in \mathbf{C}(C,D)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f \ ;$$

   ii. for any $A \in |\mathbf{C}|$, there exists a unit morphism $id_A \in \mathbf{C}(A,A)$ which is such that for any $f \in \mathbf{C}(A,B)$ we have

$$f = f \circ id_A = id_B \circ f.$$

Another way of writing $f \in \mathbf{C}(A,B)$ which also will be used is $A \xrightarrow{f} B$. Furthermore, throughout this project composed morphisms will always be of appropriate type. That is, a composition of type $h \circ g \circ f$ is only allowed if the codomain of morphism f matches up with the domain of morphism g which in turn has to have a codomain that matches the domain of morphism h, for example $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$.

The simplest category which can be depicted following these axioms is the category with only one object and one morphism. We can depict this category as follows:

$$\begin{array}{c} id_* \\ \circlearrowright \\ * \end{array}$$

, where the arrow id_* represent the unit morphism and where $*$ represent the object.

Objects and morphisms in category theory are very general concepts. The reason for this is that category theory is concerned about relations between objects rather than the objects themselves. Indeed, in the above picture it does not really matter what they are. The only important thing is that id_* is something that leaves $*$ 'unchanged' after it has acted upon it. However, we will not always operate within this "abstract world". In this project, several different *types* of categories will be defined. In some of these categories the objects and morphisms are 'identified' with familiar mathematical constructs while in others they are 'identified' with real-world systems and processes.

The following category we define is an important one, and will be used throughout this project. **Set** is the category with:

1. all sets as objects,

2. all functions between sets as morphisms,

[3] f "and then" g.

3. function composition as morphism composition, and,
4. identity functions as unit morphisms.

Since function compositions is associative and for any function $f : X \longrightarrow Y$ we have $(id_y \circ f)(x) = f(x) = (f \circ id_x)(x)$ this is indeed a category.

As stated in the introduction, the axioms of category theory made it possible for category theorists to unite mathematical structures within different areas of mathematics. The category **Set** is crucial in this regard since categories with mathematical structures as objects are built upon it. Indeed, the notion of a mathematical structure (or 'structured set') is a set-theoretic notion which will become apparent from the following section.

2.2 Concrete categories

Categories which have 'mathematical structures' as objects and 'structure-preserving maps' between these as morphisms are called *concrete categories.* These categories contain the mathematics that processes within physical systems are subjected to and are thus of great importance. We will give a more formal definition of a concrete category at the end of this chapter after we have defined the concept of 'functors'.

A *mathematical structure* is typically defined as a set of elements equipped with some operations and axioms. A group is an example of a mathematical structure. Indeed, a group G is a set of elements which includes

- an associative binary operation

$$- \bullet - : G \times G \longrightarrow G,$$

- an identity element, $1 \in G$ such that for any element $a \in G$ we have

$$a \bullet 1 = 1 \bullet a = a,$$

- an inverse a^{-1} for all $a \in G$ such that

$$a \bullet a^{-1} = a^{-1} \bullet a = 1.$$

Functions on G which preserves at least part of this structure are called *structure-preserving maps.* A group homomorphism is an example of this kind of map which is a function that preserves group multiplication.

A vector space is another example of a mathematical structure. A vector space over a field (V,$\mathbb{K}$) is an Abelian group under the law of composition specified by the operation of addition together with a field which interacts with the vectors via the operation of scalar multiplication such that $- \bullet - : V \times \mathbb{K} \rightarrow \mathbf{V}$. This scalar multiplication is further subjected to certain axioms. A liner map, a function from a vector space to another that preserves linear combinations of vectors, is then another example of a structure-preserving map.

It is now straightforward, following the axioms of category theory, to make these mathematical structures and structure-preserving maps into objects and morphisms within the appropriate category:

Grp is the concrete category with

1. all groups as objects,
2. all group homomorphisms between these groups as morphisms,
3. function composition as morphism composition, and,
4. identity functions as unit morphisms.

Indeed, this satisfy the category theory axioms since the composition of group homomorphisms is associative and identity functions are group homomorphisms.

Similarly, $\mathbf{FdVect}_{\mathbb{K}}$ is the concrete category with:

1. all finite dimensional vector spaces over the field $\mathbb{K}$ as objects,
2. all linear maps between these vector spaces as morphisms,
3. function composition as morphism composition, and,
4. identity functions as unit morphisms.

This is also a category since the composition of linear maps is associative and because identity functions are linear maps.

A related category to $\mathbf{FdVect}_{\mathbb{K}}$ which we will consider throughout this project is **FdHilb** which instead of finite dimensional vector spaces over the field $\mathbb{K}$ have finite dimensional Hilbert spaces over the complex field as objects. Hilbert spaces come with extra structure in the form of an 'inner product' which we will account for below after we introduced the concept of 'functors'.

Other examples of mathematical structures and structure-preserving maps between them which can be abstracted into concrete categories include:
Ab: which have all Abelian groups as objects and all homomorphisms as morphisms between them,
Rng: which have all rings as objects and all ring morphisms as morphisms between them,
Top: which have all topological spaces as objects and all continuous maps as morphisms between them,
Pos: which have all partially ordered sets as objects and all order preserving maps as morphisms between them,
Mon: which have all so-called *monoids*, which essentially are groups without inverses, as objects and all monoid morphisms as morphisms between them,
Rel: which have all sets as objects and all so-called *relations*, which essentially are multivalued functions, as morphisms between them, and,
Cat: which have all categories as objects and all so-called *functors*, which essentially are maps between categories, as morphisms between them.

We will explore the concept of monoids in more detail in section 2.4, where we show how to conceptualize monoids in categorical terms and how they relate to monoidal categories. The category **Cat** will be investigated more carefully in chapter 5 while the concept of *functors* will be rigorously developed in section 2.5 below. We will also return to the category **Rel** in chapter 4 where we give a proper definition of 'relations'.

Since all these categories only makes explicit reference to its objects and morphisms it may seem as if we lost the elements which constitutes the mathematical structures we started with! Fortunately this happens not to be the case. The way to recover the elements of a particular mathematical structure is to consider the morphisms within the category. In the case of the category **Set**, we can recover all elements of any set $X \in |\mathbf{Set}|$ by considering functions of the following type

$$T_x : \{*\} \longrightarrow X :: * \longmapsto x,$$

where T_x maps the element of any one-element set $\{*\}$ to a particular value $x \in X$. If X contains n elements then all of them are encoded within the hom-set $\mathbf{Set}(\{*\}, X)$.

Similarly, for any vector space $V \in |\mathbf{FdVect}_{\mathbb{K}}|$ we can single out any particular vector $v \in V$ by considering the linear map

$$T_v : \mathbb{K} \longrightarrow V :: 1 \longmapsto v,$$

where T_v maps the basis element 1 of the one-dimensional vector space $\mathbb{K}$ (which is the field itself) to a particular vector $v \in V$. All vectors in the vector space V will again be encoded within the hom-set $\mathbf{FdVect}_{\mathbb{K}}(\mathbb{K}, V)$. It suffices to map the basis element in $\mathbb{K}$ since T_v is linear.

2.3 Real-world categories

Since the axioms of category theory are so general we can construct other types of categories as well. In this section we discuss *real-world categories*. These categories have *physical systems* as objects and *processes* relevant therein as morphisms. Thus, real-world categories specifies processes within some physical system which we want to study using category theory. For example:

The real-world category **PhysProc** is the category with

1. all physical systems A,B,C,... as objects,
2. all physical processes which turns a physical system, say A, into another physical system, say B, as morphisms (these processes typically require time to be completed),
3. composition of physical processes as morphism composition, and,
4. processes which leave a physical system unchanged as unit morphisms.

This particular real-world category is very general but similar categories with modifications can easily be constructed. For example, the real-world category **QuantOpp** contains the same information as **PhysProc** but has quantum systems as objects and processes regarding these systems as morphisms instead. **QuantOpp** is moreover more suitable for processes which can be performed in an experimental set-up. Examples of processes in **QuantOpp** are preparation procedures of type $\mathrm{I} \longrightarrow Q$ where I stands for 'unspecified' while Q stands for 'qubit'. Indeed, in experiments regarding qubits in which preparation procedures are implemented, how the qubit was prepared in a certain state bears no relevance for the reminder of the experimental procedure.

Since qubits are typically mathematically modelled by two-dimensional Hilbert spaces, the real-world category **QuantOpp** and the concrete category **FdHilb** are obviously related. More precisely, processes in **QuantOpp** are mathematically modelled by morphisms in **FdHilb**. For example, the preparation procedure above is mathematically modelled by a certain morphism in **FdHilb** which singles out a particular vector (state) in a two-dimensional Hilbert space in the manner explained at the end of the previous section.

Other real-world categories within different scientific areas such as biology, chemistry, logic, computer science or engineering can also be created in a similar way. Moreover, within all real-world categories the associativity regarding composition of morphisms takes on a physical interpretation.

Indeed, given three composed processes $h \circ g \circ f$ of appropriate type, it does not matter which of those we consider as a compound entity, that is

$$h \circ g \circ f := h \circ (g \circ f) = (h \circ g) \circ f$$

We refer to this property as processes being *strict*.

2.4 Abstract categorical structures

As we mentioned regarding concrete categories, a mathematical structure is typically defined over a set. However, within the framework of category theory, since the collection of objects in any category **C** constitutes a class rather than a set, mathematical structures are rather defined over a class. This means that we can regard certain categories as being mathematical structures themselves. Indeed, this can be achieved by adding 'additional structure' or certain 'properties' to categories in order to construct so-called *abstract categorical structures*. We will give examples of two different *types* of abstract categorical structures which we will call *abstract categories* and *monoidal categories* respectively. In order to avoid confusion we will introduce these concepts one by one and from the examples and definitions explain what we mean by 'additional structure' and 'properties'. For a more rigorous approach to abstract categorical structures we refer to [18].

2.4.1 Abstract categories

Before we introduce our first example of an abstract categorical structure, we first present some examples of categories which in contrast to concrete categories are categories where the objects and morphisms are not mathematical structures (in the set-theoretic sense) and structure-preserving maps. For example:

The category $\mathbf{Mat}_{\mathbb{K}}$ is the category with

1. the set of all natural numbers $\mathbb{N}$ as objects,

2. all $m \times n$-matrices with entries in $\mathbb{K}$ as morphisms of type $n \to m$,
3. matrix composition as morphism composition, and,
4. identity matrices as unit morphisms.

We will use this category when we introduce the notion of 'functors' below.

Another example is the notion of a *monoid*. A monoid is a semigroup, that is, a group without inverses $(M, \bullet, 1)$. We can equivalently define a monoid to be a category $\mathbf{M}$ which only includes a single object $*$ in which

1. the elements of the monoid are contained within the hom-set[4] $\mathbf{M}(*, *)$,
2. the associative multiplication in the monoid is represented by the composition operation:

$$- \circ - : \mathbf{M}(*, *) \times \mathbf{M}(*, *) \longrightarrow \mathbf{M}(*, *)\,, \text{ and,}$$

3. the unit in the monoid is represented by the the unit morphism:

$$id_* \in \mathbf{M}(*, *)\,.$$

This is an example of a category in which the morphisms are so-called '*arrows*'. It is similar to the category we defined at the beginning of this section, the one with only one object and one morphism. Categories of this kind are of the abstract type, where objects and morphisms are abstracted in a manner in which specifying the object $*$ does not give any valuable information. Indeed, the important thing is not the object itself but the arrows. In particular, regarding the monoid, since there is only one object, all arrows are composable since the end of all of them coincide with the beginning of them. This is all that is needed in order to satisfy the axioms of category theory and thus $\mathbf{M}$ is indeed a category.

One way of thinking about a monoid is by imagining that the object $*$ is the set of all natural numbers $\mathbb{N}$. The morphisms can then be imagined as being functions of the following type $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for every element in the monoid, there exists a function of this type.

From the definition of the monoid $\mathbf{M}$ it follows that, for any category $\mathbf{C}$ and any $A \in |\mathbf{C}|$, the hom-set $\mathbf{C}(A, A)$ *always* constitutes a monoid.

In a similar way, any group can also be considered to be a category $\mathbf{G}$. But since groups includes inverses, any group is actually an example of an abstract category. However, before we give the definition of a group as an abstract category, we must first explain how isomorphisms works within category theory.

An isomorphism is usually thought of as the set-theoretical notion of a bijection. In category theory, the notion of a isomorphism is more general since the category $\mathbf{Set}$ is only one of many categories. However, for the category $\mathbf{Set}$, the notion of a isomorphism coincides with the usual notion of a bijection: Given functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ in $\mathbf{Set}$ where $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$, we have that if:

[4] For example, if the monoid constitutes a semigroup with four elements, one of which being the identity element, then this hom-set contains four morphisms, one of which is the identity morphism.

- $f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow x_1 = x_2$ then f is injective, and,
- for all $y \in Y$, setting $x := g(y)$, we have $f(x) = y$ then f is surjective.

Since the converse also holds, f is indeed a bijection. This type of reasoning generalises to all concrete categories since they have mathematical structures (structured sets) as objects. A more general definition is given by:

Definition (Isomorphism). Two objects $A, B \in |\mathbf{C}|$ are isomorphic if there exists two morphisms, $f \in \mathbf{C}(A,B)$ and $g \in \mathbf{C}(B,A)$ such that $g \circ f = id_A$ and $f \circ g = id_B$. The morphism f is then called an isomorphism while $g := f^{-1}$ is called the inverse to f.

Since a group $(G, \bullet, 1)$ is a monoid with inverses we can now alternatively define a group to be an abstract category $\mathbf{G}$ which has one element $*$ in which

1. the elements of the group are contained within the hom-set $\mathbf{G}(*,*)$,
2. the associative multiplication in the group is represented by the composition operation:
$$- \circ - : \mathbf{G}(*,*) \times \mathbf{G}(*,*) \longrightarrow \mathbf{G}(*,*)\,, \text{ and,}$$
3. the unit in the group is represented with the unit morphism:
$$id_* \in \mathbf{G}(*,*)\,.$$

   - Additional property:

All morphisms are isomorphisms.

From this definition we also have that in any category $\mathbf{C}$, for any $A \in |\mathbf{C}|$, the hom-set $\mathbf{C}(A,A)$ constitutes a group *if and only if* the morphisms are isomorphisms. More generally, any category $\mathbf{C}$ in which, for any objects $A,B \in |\mathbf{C}|$ the hom-set $\mathbf{C}(A,B)$ is a set of isomorphisms, is called a *groupoid*.

Another example of an abstract category involves the concept of any preordered set $(P, \leq)$ in which the elements satisfy:

- Reflexivity,

For any $a \in P$ we have: $a \leq a$.

- Transitivity,

For any $a,b,c \in P$ we have: if $a \leq b$ and $b \leq c$ then $a \leq c$.

Reflexivity and transitivity are thus examples of properties which specify how elements relates to themselves and other elements respectively. Now, since morphisms in category theory are an abstracted notion of relations, we can alternatively define any preordered set to be an abstract category $\mathbf{P}$ in which

1. the elements of P are the objects of $\mathbf{P}$,

2. whenever $a \leq b$ for any $a,b \in P$ then the hom-set $\mathbf{P}(a,b)$ is a singleton, that is, there exist only one morphism between a and b, and if $a \nleq b$ then the hom-set $\mathbf{P}(a,b)$ is empty,

3. whenever $a \leq b$ and $b \leq c$, then transitivity ensures that the hom-set $\mathbf{P}(a,c)$ is a singleton[5] which we regard as composition in $\mathbf{P}$, and,

4. reflexivity assures that the hom-set $\mathbf{P}(a,a)$ is a singleton which we regard as the unit morphism in $\mathbf{P}$.

- Additional property:

Every hom-set contains at most one morphism.

From this definition it follows that for any category $\mathbf{C}$ in which the objects constitutes a set and in which all hom-sets are either singletons or empty, is in fact a preorder set. Such categories are called *thin* categories while categories with non-trivial hom-sets are called *thick*.

As we have seen from these examples, categories with some additional property added to the morphisms are examples of abstract categorical structures which we call abstract categories. Note in particular that the category $\mathbf{M}$, which is a monoid within the framework of category theory, does not constitute an abstract category from the definition we gave. No additional properties where needed in order to state that definition which is why the category $\mathbf{M}$ does not constitute an abstract category. However, in the following section we will give an equivalent definition of a monoid in which we do need an additional property. Thus, a monoid can be conceived, within the framework of category theory, as a simple category and/or an abstract category depending on the definition.

2.4.2 Monoidal categories

In contrast to abstract categories, we can construct a different type of abstract categorical structure by adding 'additional structure' to a category $\mathbf{C}$. And just as there are many examples of abstract categories, there are many examples of monoidal categories. However, contrary to abstract categories, these are not constructed by changing the additional constraint. Indeed, all monoidal categories comes with the same additional structure, namely the *monoid structure* $(|\mathbf{C}|, \otimes, \mathrm{I})$. They are instead obtained by changing the *tensor structure*. The notion of tensor structure will become apparent from the discussion after the following definition of a *strict monoidal category* which is a very important example of a monoidal category:

Definition (Strict monoidal category). A *strict monoidal category* is a category $\mathbf{C}$ which moreover comes with additional structure provided by the associative operation $\otimes$ and a unit object I such that

1. for any $A, B, C \in |\mathbf{C}|$,

$$\otimes (B \otimes C) = (A \otimes B) \otimes C \text{ and } \mathrm{I} \otimes A = A = A \otimes \mathrm{I}, \tag{2.1}$$

A

[5]The word 'singleton' simply refers to the fact that there is only one morphism within the hom-set $\mathbf{P}(a,b)$.

2. for any objects A,B,C,$D \in |\mathbf{C}|$ there exists an operation,

$$- \otimes - : \mathbf{C}\,(A,B) \times \mathbf{C}\,(C,D) \longrightarrow \mathbf{C}\,(A \otimes C, B \otimes D) :: (f,g) \longmapsto f \otimes g \tag{2.2}$$

which is associative and has id_{I} as unit morphism[6], that is,

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h \;\text{ and }\; id_{\mathrm{I}} \otimes f = f = f \otimes id_{\mathrm{I}},$$

3. for any morphisms $f, g, h, k \in \mathbf{C}$ of appropriate type we have

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)\,, \tag{2.3}$$

4. for any objects $A, B \in |\mathbf{C}|$ we have

$$id_A \otimes id_B = id_{A \otimes B}. \tag{2.4}$$

The strict monoidal category is crucial since it explicitly axiomatizes the common features which are shared by real-world categories. In other words, the strict monoidal category is a generic structure *onto which real-world categories fit.* By appropriately identifying systems and processes within some particular real-world category with the more abstract notions of objects and morphisms within the strict monoidal category, we can use the mathematics of category theory in order to get a better understanding regarding them.

The associative operation $\otimes$ is called the *monoidal product* and enables manipulation of two (or more) objects at the same time. Morphisms within strict monoidal categories can thus map two objects to two different objects by axiom two, 2.2. Note that this is in contrast to the concrete categories which only considers one object at a time[7]. Moreover, by viewing real-world categories as strict monoidal categories, the monoidal product can be interpreted as a logical conjunction in which:

- $A \otimes B :=$ system A <u>and</u> (while) system B
- $f \otimes g :=$ process f <u>and</u> (while) process g.

Axiom three, 2.3, is a property called the *interchange law.* In fact, all monoidal categories shares this property and we present a proof of it below in section 2.5. By viewing real-world categories as strict monoidal categories, the interchange law explicitly states that, given morphisms of appropriate type, it does not matter whether we regard:

- (process f and then process g) while (process h and then process k) or
- (process f and process h) and then (process g and process k).

The unit object I is formally called the *monoidal unit* and is defined through axiom one, 2.1, which, if we consider real-world categories as strict monoidal categories, explicitly state that

[6] The associative composition of morphisms in any monoidal category is analogous to the associative composition of elements in a monoid, hence the name strict 'monoidal' category.

[7] For this reason, it is common to refer to concrete categories as being 'one-dimensional' while monoidal categories are referred to as 'two-dimensional'.

- I is any system in which any other system $A \in |\mathbf{C}|$ is left invariant when adjoined to it.

We gave an example regarding the real-world category **QuantOpp** above where the monoidal unit was referred to as 'unspecified'. It is also sometimes called 'no system' or 'nothing'. Again, within the context of real-world categories as strict monoidal categories, the morphism id_{I} corresponds to a process in which nothing is done to nothing. Axiom four, 2.4, thus explicitly states that

- doing nothing to system A while doing nothing to system $B :=$ doing nothing to system A and B.

Of course, all four of these axioms are rather self explanatory, however when dealing with physical systems and processes categorically, we first need to establish a firm ground on which we then may expand. Indeed, in the next chapter we will show how we may expand upon the strict monoidal category such that more specific real-world categories can be considered.

Incidentally, this type of axiomatization, in which we prescribe rules for how the objects and morphisms must behave with regards to the monoidal product and the monoidal unit is referred to as *tensor structure*. From a categorical viewpoint, the particular tensor structure given by the definition of the strict monoidal category constitutes a universal axiomatization for all real-world categories. However, most real-world categories include processes which are more involved and thus not accounted for by this simple tensor structure.

For example, in the introduction we mentioned a real-world category involving anyons for which a more involved tensor structure constitutes the axiomatization. Indeed, in order to capture processes like 'the braiding of anyons' we must modify the tensor structure for the strict monoidal category by adding additional axioms to it in an appropriate and consistent way. This is something which we will explain how to do in the next chapter albeit more generally. However, these are the kinds of real-world categories we truly care about!

Now, a strict monoidal category is not only a generic structure onto which real-world categories fit. Indeed, any category that fits onto the structure of a strict monoidal category is in fact a strict monoidal category. A case in point is the monoid $(M, \bullet, 1)$, which we discussed in the last subsection. However, in order to make the monoid into a strict monoidal category we first need to show how to establish it as an abstract category. This can be achieved by identifying

1. the elements M of the monoid with with objects in a category $\mathbf{M}$,
2. the associative multiplication of the monoid with the monoidal product $\otimes \in \mathbf{M}$, and,
3. the unit 1 of the monoid with the monoidal unit $\mathrm{I} \in \mathbf{M}$

    - Additional property:

All morphisms are identity morphisms.

Since we explicitly add a property on morphisms, in this case that all morphisms are identity morphisms, the monoid indeed constitutes an abstract category. Now, by equipping the identity morphisms with the same monoid structure as we have for the objects we satisfy all the axioms given for the strict monoidal category. For example, since all morphisms are identity morphisms, axiom three for the strict monoidal category is satisfied since

$$(id_A \circ id_A) \otimes (id_B \circ id_B) = id_A \otimes id_B = id_{A\otimes B} = id_{A\otimes B} \circ id_{A\otimes B} = (id_A \otimes id_B) \circ (id_A \otimes id_B).$$

Thus, a monoid can equivalently be conceived as a category with only one object $*$ where the morphisms are identified with the elements, or an abstract category where the additional property is that all morphisms are identity morphisms, or as a strict monoidal category in which the identity morphisms are equipped with monoid structure! However, this is by no means unique. Indeed, as we will see in subsequent chapters, many categories can in fact be considered to be different types of categories simultaneously. To avoid confusion we will refer to categories by their type whenever an ambiguity may be present so that the distinction between, for example, the category **M** and the abstract category **M** becomes clear.

Thus, from the examples and discussions regarding abstract categorical structures, the fundamental distinction between abstract categories and monoidal categories is that

- Abstract categories are categories in which some property on morphisms is explicitly added, for example, the property that all morphisms are isomorphisms as is the case for groups.
- Monoidal categories are categories in which the morphisms are equipped with monoid structure, that is, morphisms in monoidal categories are subjected to the same axioms as the elements in a monoid are.

2.5 Functors

Above we have mentioned three kinds of categories, namely:

- *Concrete categories* which have mathematical structures as objects and structure preserving maps between them as morphisms.
- *Real-world categories* which have physical systems as objects and processes thereon as morphisms.
- *Abstract categorical structures* which are mathematical structures with additional structure or properties added to them.

As we have already alluded to, these types of categories are all relevant in order to use *categories as physical models*. In this regard, a real-world category specifies a system of some area we wish to study, which for example can be quantum theory, computation, biology, chemistry or proof theory. The mathematics governing that system is contained in some concrete category in which, in the case of quantum theory, the objects are Hilbert spaces and the morphisms are linear maps. Abstract categorical structures of monoidal type are then used for axiomatization, that is, we can study what tensor structure gives rise to certain physical phenomena, for example the braiding of anyons [6].

However, the relationship between these types of categories has not been made explicit yet. Within category theory the relationship is made rigorous by so-called *functors*. These are maps between categories that preserves the *structure of categories*, that is, composition and identities. In order to achieve this we define a functor as follows:

Definition (Functor). Let $\mathbf{C}$ and $\mathbf{D}$ be any types of categories. A *functor*

$$F : \mathbf{C} \longrightarrow \mathbf{D},$$

is then a function consisting of two kinds of mappings: one map on the objects, and a family of maps on the hom-set, that is:

1. For any $A \in |\mathbf{C}|$, a map on objects:

$$F : |\mathbf{C}| \longrightarrow |\mathbf{D}| :: A \longmapsto FA,$$

2. For any $A, B \in |\mathbf{C}|$, a family of maps on hom-sets:

$$F : \mathbf{C}\,(A,B) \longrightarrow \mathbf{D}\,(FA, FB) :: f \longmapsto Ff,$$

which preserves composition[8] and identities, that is:

i. for any $f \in \mathbf{C}\,(A, B)$ and $g \in \mathbf{C}\,(B, C)$ we have

$$F\,(g \circ f) = Fg \circ Ff, \text{ and,}$$

ii. for any $A \in |\mathbf{C}|$ we have

$$Fid_A = id_{FA}.$$

This is indeed a important concept, for say we wish to mathematically model the real-world category **PhysProc** with some concrete category, **Mod**. We can then use a functor to explicitly assign to each process in the real-world category a morphism in the concrete category:

$$F : \mathbf{PhysProc} \longrightarrow \mathbf{Mod}.$$

Moreover, since we use a functor to perform this mapping, composition of processes and identity processes in the real-world category will automatically by preserved as composition of morphisms and unit morphisms respectively in the concrete category. Thus, functors can be used as a "link" between the system we want to study and the mathematics that system obeys.

However, this is not the only way we can use functors. Since we can map between any types of categories, other kinds of maps are also possible. For example, say we pick some basis for all vector spaces $V \in |\mathbf{FdVect}_{\mathbb{K}}|$ so that any linear function $f \in \mathbf{FdVect}_{\mathbb{K}}\,(V, W)$ admits a matrix in these bases. Within category theory this assignment of bases is specified through a functor

$$F : \mathbf{FdVect}_{\mathbb{K}} \longrightarrow \mathbf{Mat}_{\mathbb{K}}$$

where $\mathbf{Mat}_{\mathbb{K}}$ is the category we defined above with all natural numbers as objects and matrices with entries in $\mathbb{K}$ as morphisms.

[8]this property is formally referred to as *functoriality*.

Now, lets define the category $\mathbf{Mat}_{\mathbb{C}}$ which contains all the information $\mathbf{Mat}_{\mathbb{K}}$ does but where the matrices are transposed. The way we map between these categories so that this is accounted for is by using a *contravariant functor*, which contains the same information as ordinary functors do up to reversal of composition, that is:

Definition (Contravariant functor). A *contravariant functor* $F' : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor that reverses composition:

$$F' (g \circ f) = F' f \circ F' g.$$

The map, $F' : \mathbf{Mat}_{\mathbb{K}} \longrightarrow \mathbf{Mat}_{\mathbb{C}}$, will then preserve objects and identities but reverse matrices and matrix composition which is what we want.

The category $\mathbf{Mat}_{\mathbb{C}}$ is moreover called the *opposite category* of $\mathbf{Mat}_{\mathbb{K}}$ since they only differ by the reversal of morphisms. More generally we have that:

Definition (Opposite category). The *opposite category* $\mathbf{C}^{op}$ of a category $\mathbf{C}$ is the category with

- the same objects as $\mathbf{C}$,
- reversed morphisms relative to $\mathbf{C}$, that is,

  $$f \in \mathbf{C}(A, B) \Rightarrow f \in \mathbf{C}^{op}(B, A),$$

  where we will denote $f \in \mathbf{C}^{op}(B, A)$ by f^{op} to avoid confusion.
- the same identity morphisms as in $\mathbf{C}$, and,
- composition being reversed,

  $$(g \circ f)^{op} = f^{op} \circ g^{op}.$$

Any morphism f^{op} is moreover involutive, that is, reversing the arrow twice is the same thing as doing nothing.

We now have all the information needed to define the concrete category **FdHilb** which has finite dimensional Hilbert spaces as objects and linear maps as morphisms. This category is crucial for the implementation of category theory to quantum theory and will be used throughout this project.

A Hilbert space is a vector space over the complex field $\mathbb{C}$ which includes an inner product. The inner product, in the case of Hilbert spaces, is defined as a map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ which assigns to each ordered pair $\mathbf{u}$, $\mathbf{v} \in \mathcal{H}$ a complex number $\langle \mathbf{u}, \mathbf{v} \rangle$ with the following properties:

- Hermiticity,

  $$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$$
- Linearity,

  $$\langle \mathbf{w}, \alpha \mathbf{u} + \beta \mathbf{v} \rangle = \alpha \langle \mathbf{w}, \mathbf{u} \rangle + \beta \langle \mathbf{w}, \mathbf{v} \rangle$$

- Positivity,
$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$$
with equality if and only if $\mathbf{u} = \mathbf{0}$.

In order to account for the inner product in term of category-theoretical concepts we now define the *hermitian transpose* which is an involutive contravariant functor (or involutive endofunctor),
$$\dagger : \mathbf{FdHilb}^{op} \longrightarrow \mathbf{FdHilb},$$
which

1. is *identity-on-object (self-duality)*, that is,
$$\dagger : |\mathbf{FdHilb}^{op}| \longrightarrow |\mathbf{FdHilb}| :: \mathcal{H} \mapsto \mathcal{H},$$
2. assigns morphisms to their adjoints, that is,
$$\dagger : \mathbf{FdHilb}^{op}(\mathcal{H}, \mathcal{K}) \longrightarrow \mathbf{FdHilb}(\mathcal{K}, \mathcal{H}) :: f \mapsto f^{\dagger}.$$

This indeed satisfy the definition of a contravariant functor since
$$id_{\mathcal{H}}^{\dagger} = id_{\mathcal{H}}$$
and for $f \in \mathbf{FdHilb}(\mathcal{H}, \mathcal{K})$ and $g \in \mathbf{FdHilb}(\mathcal{K}, \mathcal{L})$ we have
$$(g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}.$$

With the hermitian transpose defined, we have been able to specify the adjoint which, as we will see in the next chapter, will be what we need to recover the inner product. Incidentally, self-duality in the form of involutive endofunctors on categories has been considered as early as 1950 [24]. A link between adjoint functors and adjoints in Hilbert spaces was first made precise in 1974 [25]. The exploitation of daggers, which we will elaborate on in subsequent chapters, originated with Selinger in 2008 [26].

With functors now defined, we can give a more formal definition of a concrete category:

Definition (Concrete category). A *concrete category* is a category $\mathbf{C}$ together with a functor $U : \mathbf{C} \longrightarrow \mathbf{Set}$, that is, a pair $(\mathbf{C}, U)$.

The functor U is sometimes called the 'forgetful functor' since it sends the mathematical structure of the concrete category to the underlying set, and morphisms to the underlying functions. For example, the category **Grp** is a concrete category for the following forgetful functor:
$$U : \mathbf{Grp} \longrightarrow \mathbf{Set} :: \begin{cases} (G, \bullet, 1) \longmapsto G \\ \quad f \longmapsto f \end{cases}$$
which "forgets" the groups multiplication and unit.

More generally, a **D**-concrete category is a category **C** in which the underlying structure is mapped to via the forgetful functor $U : \mathbf{C} \longrightarrow \mathbf{D}$.

Moreover, the law of interchange added to the strict monoidal category above can now be proven. The reason we prove it here is that the monoidal product $\otimes$ is in fact a functor. Indeed, $\otimes$ is both a map on objects and on hom-sets, that is,

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}.$$

Recall that the interchange law states that

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h).$$

Proof (Bifunctor).

$$\begin{aligned} (g \circ f) \otimes (k \circ h) &:= \otimes (g \circ f, k \circ h) \\ &= \otimes ((g,k) \circ (f,h)) \quad \text{(definition of } \mathbf{C} \times \mathbf{C}) \\ &= (\otimes(g,k) \circ \otimes(f,h)) \quad \text{(funtoriality of } \otimes) \\ &= (g \otimes k) \circ (f \otimes h) \end{aligned} \tag{2.5}$$

Recall the functoriality property for a functors which states that $F(g \circ f) = Fg \circ Ff$. ■

With this proof we conclude this chapter. We have developed a number of relevant notions of category theory which we will use in order to modify the tensor structure of the strict monoidal category such that more specific real-world- physical systems and processes can be axiomatized. In particular, we will show how the hermitian transposed defined above can be incorporated into the tensor structure of the strict monoidal category such that general processes within quantum systems can be specified purely in categorical language. This will moreover enable us to differentiate between families of real-world categories which have common processes, in particular, how quantum processes differ from classical processes.

However, in order to use category theory to get physical insight into real-world categories it does not suffice to modify the tensor structure of the strict monoidal category! The reason for this is that the objects and morphisms within strict monoidal categories represent real-world physical systems which evolve due to real-world processes and not the mathematical representations of these. Indeed, since physicists use mathematics in order to represent the physical world we need to construct categories which axiomatizes the underlying mathematics of real-world categories, not the real-world categories themselves.

Thus, keeping the underlying real-world category in mind, we need to define a tensor structure onto which relevant concrete categories can fit. This will be the main topic of concern in the next chapter.

3

Categorical Axiomatization of Physical systems

In order to build real-world models out of category theory we must construct a category with a tensor structure which axiomatizes the particular physical phenomena (phenomenons) we want to study. But since the variety of processes regarding physical systems is vast, any one category cannot be dynamic enough to accommodate this. Instead we solve this issue by constructing a general category, which however unspecified, at least incorporates a tensor structure that all physical systems are expected to share. This is the reason why we introduced the strict monoidal category in the last chapter. In this chapter we show how to expand upon the set of axioms given by the tensor structure of the strict monoidal category such that axiomatization of more elaborate processes within real-world categories can be specified. However, in order to *construct physical models from category theory* this is not enough. Indeed, in order to really get physical insight into processes within physical systems we need to axiomatize the underlying mathematics governing them. And since the mathematics of real-world categories are contained in concrete categories, and axiomatization is constructed from monoidal categories, we will take the same approach as above and construct a monoidal category that has a general enough tensor structure to accommodate most mathematical structures that are used for the study of real-world categories. That is, keeping real-world categories in mind, we will construct a monoidal category onto which concrete categories fit. We will also show how the tensor structure regarding this *non-strict* monoidal category can be modified such that more specific mathematics can be axiomatized.

Thus, the fundamental difference between strict monoidal categories and non-strict monoidal categories (or simply monoidal categories) is

- Strict monoidal categories:

  Axiomatizes rules for which real-world- physical systems and processes obeys.

- Monoidal categories:

  Axiomatizes rules for which the underlying mathematics of real-world- physical systems and processes obeys.

3.1 Strict symmetric tensor structure

In the last chapter we defined the strict monoidal category which includes the monoidal product $\otimes$ which enable us to manipulate two (or more) objects simultaneously. We can thus construct real-world categories such as **CQOpp** where

- objects are all classical <u>and</u> quantum systems, and,
- morphisms are operations thereon.

Now, say we are interested in a process that involves both a classical system and a quantum system

$$X \otimes Q \xrightarrow{f\otimes h} X' \otimes Q',$$

that is, we start with some classical system and some quantum system $X \otimes Q$, perform two operations, one on the classical system and the other one on the quantum system $f \otimes h$ such that we end up with a different classical system and a different quantum system $X' \otimes Q'$.

Intuitively, we know that the process

$$Q \otimes X \xrightarrow{h\otimes f} Q' \otimes X',$$

is identical to the above. However, within the framework of category theory we cannot simply equate them. The reason for this is that morphisms between any two objects in category theory are elements within the corresponding hom-set. If it where true that

$$f \otimes h = h \otimes f$$

then this would imply that $f \otimes h$ and $h \otimes f$ would live in the same hom-set which further would imply that

$$X \otimes Q = Q \otimes X \quad \text{and} \quad X' \otimes Q' = Q' \otimes X'$$

must be true. But the problem with this is that we no longer can distinguish between the classical system and the quantum system!

The solution to this issue is obtained by introducing an operation called *symmetry*,

$$\sigma_{X,Q} : X \otimes Q \longrightarrow Q \otimes X,$$

which explicitly interchange the role of the classical system and quantum system relative to the monoidal product. The intuition we mentioned above can thus be stated within the context of category theory by the following, defining equation

$$\sigma_{X',Q'} \circ (f \otimes h) = (h \otimes f) \circ \sigma_{X,Q}. \tag{3.1}$$

In order to make reference to the initial objects more explicitly, this equation can alternatively be expressed as a *commutative diagram*:

$$\begin{array}{ccc} X \otimes Q & \xrightarrow{\sigma_{X,Q}} & Q \otimes X \\ {\scriptstyle f\otimes h}\downarrow & & \downarrow{\scriptstyle h\otimes f} \\ X' \otimes Q' & \xrightarrow[\sigma_{X',Q'}]{} & Q' \otimes X' \end{array} \tag{3.2}$$

Since equation 3.1 is a defining equation for symmetries, this commutative diagram is called a *naturality condition*. Naturality conditions are constructed in order to make sure that any additional morphism-structure (in this case symmetry) is well behaved, that is, commutes with the objects and morphisms already established within the category. Moreover, if several morphisms are added, then they must commute with each other as well. Commutative diagrams constructed in order to verify commutation between several morphisms are called *coherence conditions*. We will discuss naturality- and coherence conditions in greater detail in section 3.4 where we introduce non-strict monoidal categories.

With symmetries now defined, we can use them to modify the tensor structure of the strict monoidal category. More generally we have:

Definition (Strict symmetric monoidal category). A *strict symmetric monoidal category* is a strict monoidal category $\mathbf{C}$ which moreover comes with symmetries

$$\left\{ A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \mid A,B \in |\mathbf{C}| \right\}$$

such that

1. for all $A,B \in |\mathbf{C}|$ we have $\sigma^{-1}_{A,B} = \sigma_{B,A}$, and,
2. for all $A,B,C,D \in |\mathbf{C}|$ and all f,g of appropriate type we have

$$\sigma_{C,D} \circ (f \otimes g) = (g \otimes f) \circ \sigma_{A,B}.$$

The first axiom states that all symmetries are isomorphisms which is a consequence from the following equation which is a general law for any type of interchange:

$$\sigma_{B,A} \circ \sigma_{A,B} = id_{A \otimes B}.$$

Both real-world categories **PhysProc** and **QuantOpp** are examples of categories that fit onto the structure of the strict symmetric monoidal category.

3.2 Graphical calculus for strict monoidal categories

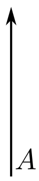
All strict monoidal categories admit a purely diagrammatic calculus. This is a very powerful tool which can substantially reduce the complexity of algebraic manipulations. It also lessens the gap between the $(\circ, \otimes)$-logic we intuitively have regarding systems and processes and the categorical structures we construct to mathematically axiomatize them. The graphical calculus within any strict monoidal category is moreover characterized by:

- There exists a graphical representation for all symbols used to define any strict monoidal category, for example $\otimes$, $\circ$, I, A, f, $\sigma_{A,B}$ and so on,
- The tensor structure for any strict monoidal category can be represented by intuitive graphical manipulations,
- If an equation is derivable from the tensor structure of any strict monoidal category then it is also derivable by graphical calculus.

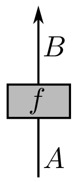
The general notion of diagrammatic reasoning was first introduced in Penrose's work in the 1970s [27]. It was formalized for monoidal categories by Joyal and Street in 1991 [28]. For a more modern survey of graphical calculus, see Peter Selinger's paper [2].

The graphical representations for the tensor structure in strict symmetric monoidal categories are given by:

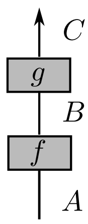
- The identity morphism id_I is not depicted at all.
- The identity morphism id_A for the object A is depicted as



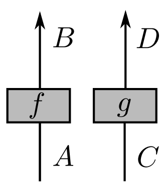
- A morphism $f : A \longrightarrow B$ is depicted as:



- The composition of morphisms $g \circ f$ with $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is depicted as:



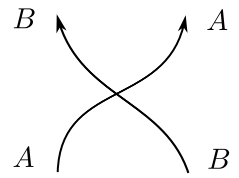
- The tensor product $f \otimes g$ with $f : A \longrightarrow B$ and $g : C \longrightarrow D$ is depicted as:



- The symmetry

$$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

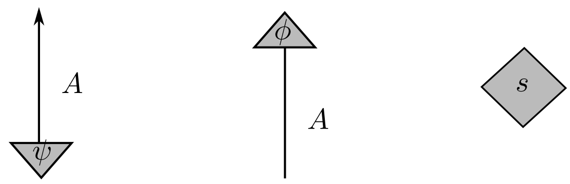
is depicted as:



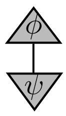
- Morphisms such as,

$$\psi : \mathrm{I} \longrightarrow A, \;\; \phi : A \longrightarrow \mathrm{I} \text{ and } s : \mathrm{I} \longrightarrow \mathrm{I}$$

are respectively depicted as:



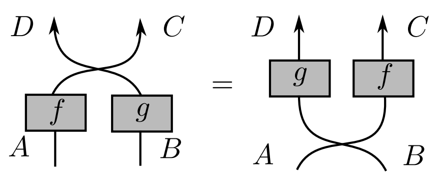
The diamond picture above is actually an aberration for the composition:



since the morphism $s = \phi \circ \psi$.

In the next section we will show how the last three morphisms we depicted can be used to graphically represent Dirac notation.

With graphical calculus now established, we can use it to represent equations. For example, the defining equation of symmetry 3.1 can equivalently be stated by the following picture:

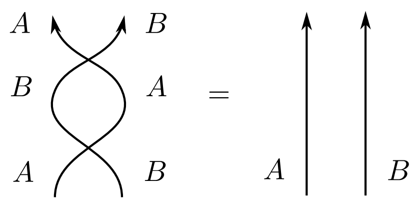


, where each side of this pictorial equation describes one of the two paths in which: $A \otimes B \to D \otimes C$ in the naturality condition 3.2.

Furthermore, the equation

$$\sigma_{B,A} \circ \sigma_{A,B} = id_{A \otimes B}$$

can also be expressed graphically:



Thus, we now have three alternative ways of expressing equations:

- Algebraic equations,
- Commutative diagrams, and,
- Graphical representations.

As we explained above, commutative diagrams in the form of naturality conditions are used to verify defining algebraic equations. They can be thought of as "consistency proofs" which are used to define morphism-structure purely in categorical terms. However, there are further situations in which diagrams can be used. Indeed, if we are interested in verifying algebraic equations within already established strict monoidal categories, a method known as *diagram chasing* may be implemented [20], [22], [23]. However, this method is often rather tedious and complicated but was historically an important step towards a better geometrical understanding of strict monoidal categories [16]. Fortunately enough, an alternative way can equivalently be implemented using graphical calculus, which is both more economic and informative (see section 4.2).

Since real-world categories fit onto the structure of strict monoidal categories, graphical representations can also be used in order to get a better geometrical understanding for processes. Indeed, graphical calculus holds great potential as a tool, and is currently used within the study of categorical quantum information theory as it gives a rigorous pictorial way of dealing with processes therein [29], [3], [19].

3.3 Strict dagger tensor structure

Above we showed how we could add symmetries to the strict monoidal category in order to axiomatize more specific real-world categories. Here we will continue the modification so that quantum systems can be more precisely axiomatized. In order to achieve this we need to specify the inner product which requires the hermititan transpose we defined in chapter 2. Generally we have:

Definition (Strict dagger monoidal category). A *strict dagger monoidal category* $\mathbf{C}$ is a strict monoidal category equipped with an involutive identity-on-objects contravariant functor

$$\dagger : \mathbf{C}^{op} \longrightarrow \mathbf{C},$$

which is such that,

1. for all $A \in |\mathbf{C}|$ we have $A^\dagger = A$.
2. for all morphisms f we have $f^{\dagger\dagger} = f$, and,

3. this functor preserves the monoidal product

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger.$$

Morphisms such as $B \xrightarrow{f^\dagger} A$ will be referred to as the adjoint of $A \xrightarrow{f} B$.

The real-world category **QuantOpp** is an example of a category which fits onto this structure while **PhysProc** does not. Indeed, as we show below, this addition of structure to the strict monoidal category will allow us to specify familiar quantum notions such as states, effects and probability weights.

The added structure we just defined can moreover be combined with the structure of the strict symmetric monoidal category:

Definition (Strict dagger symmetric monoidal category). A *strict dagger symmetric monoidal category* $\mathbf{C}$ is both a strict dagger monoidal category and a strict symmetric monoidal category such that

- for any A,$B \in |\mathbf{C}|$ the symmetry $\sigma_{A,B}$ is *unitary*, that is,

$$\sigma^\dagger_{A,B} = \sigma^{-1}_{A,B}.$$

The structure defined over the strict dagger monoidal category can now be used in order to specify the inner product. Indeed, for any two morphisms $\psi, \phi : \mathrm{I} \longrightarrow A \in \mathbf{C}$, their inner product is defined as

$$\langle \phi \,|\, \psi \rangle := \phi^\dagger \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}.$$

These kinds of morphisms where we map from the monoidal unit to itself are referred to *scalars*.

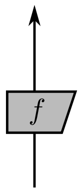
Moreover, using unitary morphisms together with the above definition of the inner product, the dagger functor of the strict dagger monoidal category can also be used to specify the defining property of adjoints and unitary matrices strictly in terms of categorical concepts. Indeed, for the adjoint we have

$$\begin{aligned} \langle f^\dagger \circ \psi \,|\, \phi \rangle &= (f^\dagger \circ \psi)^\dagger \circ \phi \\ &= (\psi^\dagger \circ f) \circ \phi \\ &= \psi^\dagger \circ (f \circ \phi) \\ &= \langle \psi \,|\, f \circ \phi \rangle \,. \end{aligned} \tag{3.3}$$

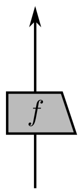
From this we can also show that unitary morphisms preserves the inner product

$$\begin{aligned} \langle U \circ \psi \,|\, U \circ \phi \rangle &= \langle U^\dagger \circ (U \circ \psi) \,|\, \phi \rangle \\ &= \langle (U^\dagger \circ U) \circ \psi \,|\, \phi \rangle \\ &= \langle \psi \,|\, \phi \rangle \,. \end{aligned} \tag{3.4}$$

Furthermore, the graphical calculus developed in the previous section also extend to encompass the structure defined for the strict dagger monoidal category. This is done by introducing an asymmetry to the morphism $f : A \longrightarrow B$ such that the graphical representation is given by

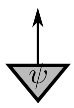
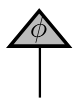
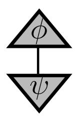
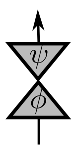


The adjoint $B \xrightarrow{f^\dagger} A$ is then depicted as



that is, we turn the box representing $A \xrightarrow{f} B$ upside-down.

From the definition of the inner product together with equations 3.3 and 3.4 we can now interpret Dirac notation in terms of strict dagger monoidal categories which moreover can be represented graphically:

Dirac	**Morphism**	**Graph**
$\lvert\psi\rangle$	$\mathrm{I} \xrightarrow{\psi} A$	
$\langle\phi\rvert$	$A \xrightarrow{\phi^\dagger} \mathrm{I}$	
$\langle\phi\vert\psi\rangle$	$\mathrm{I} \xrightarrow{\psi} A \xrightarrow{\phi^\dagger} \mathrm{I}$	
$\lvert\psi\rangle\langle\phi\rvert$	$A \xrightarrow{\phi^\dagger} \mathrm{I} \xrightarrow{\psi} A$	

In real-world categories such as **QuantOpp**, the morphisms $\psi : \mathrm{I} \rightarrow A$ and $\phi : A \rightarrow \mathrm{I}$ represent states and effects respectively, while morphisms $S : \mathrm{I} \rightarrow \mathrm{I}$ represent probabilistic weights, that is, the probability of a certain effect to occur when the system is in a particular state. These are the values which, within the framework of familiar quantum mechanics, are obtained by the Born rule.

Now, since the mathematics within **QuantOpp** is contained in the concrete category **FdHilb**, certain morphisms therein should also corresponds to the morphisms in the second column of the above table. Recall that we in chapter 2 showed that any vector in any vector space could be faithfully represented by considering linear maps of the following type

$$T_v : \mathbb{K} \longrightarrow V :: 1 \longmapsto v.$$

Similarly, we can also consider linear maps which singles out any number in the 'one-dimensional vector space' $\mathbb{K}$. This is done by considering linear maps of the following type

$$S_x : \mathbb{K} \longrightarrow \mathbb{K} :: 1 \longmapsto x.$$

Analogously, the same procedure can be applied to the concrete category **FdHilb**. In doing so we can identify these linear maps with the morphisms from the table above, that is:

$$\mathbb{C} \longrightarrow \mathcal{H} \quad \sim \quad \mathrm{I} \xrightarrow{\psi} A$$

$$\mathcal{H} \longrightarrow \mathbb{C} \quad \sim \quad A \xrightarrow{\phi^\dagger} \mathrm{I}$$

$$\mathbb{C} \longrightarrow \mathbb{C} \quad \sim \quad \mathrm{I} \xrightarrow{\psi} A \xrightarrow{\phi^\dagger} \mathrm{I}$$

Moreover, since any bra and/or ket can be represented by a corresponding matrix, we can account for this in the concrete category **FdHilb** by picking some basis for all Hilbert spaces $\mathcal{H} \in |\mathbf{FdHilb}|$ in an analogous manner to the way we did it for the concrete category $\mathbf{FdVect}_{\mathbb{K}}$ in chapter 2. However, although the concrete category **FdHilb** can be used to mathematically model states, effects and weights in this way, we cannot use it to mathematically model other relevant processes in **QuantOpp** in which more than one object is involved. This is because concrete categories does not include structure in the form of a monoidal product which is necessary in order to achieve this. In order to solve this issue we must therefore make the concrete category **FdHilb** into a *monoidal category* which we will show how to do in the following section.

3.4 Monoidal categories revisited

As we explained in the introduction of this chapter, real-world categories are axiomatised by the strict monoidal category. Now, since the mathematics of real-world categories are contained in concrete categories we also need to construct a category with a general enough structure so that mathematical structures used to model systems and processes can be accommodated. However, since concrete categories are set-theory based, the tensor structure in strict monoidal categories is no longer valid. More specifically, the following axioms do not hold for mathematical structures

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \qquad \text{and} \qquad A \otimes \mathrm{I} = A = \mathrm{I} \otimes A.$$

This is due to the fact that the underlying sets of mathematical structures are defined as ordered pairs in which the order of the elements is significant, that is:

$$(a,b) = (c,d) \text{ if and only if } a = c \text{ and } b = d.$$

This defining property can be achieved using Kuratowski's definition [30] of ordered pairs:

$$(a,b) = \Big\{ \{a\}, \{a,b\}, \Big\}$$

, where the notations (a,b) and $\{a,b\}$ represent an order pair and an unordered pair respectively. Now, using this definition it follows (see Appendix A) that, for any sets X,Y,Z, we have

$$(x,(y,z)) \neq ((x,y),z) \qquad \text{and} \qquad (x,*) \neq x \neq (*,x),$$

which has as a consequence that

$$X \times (Y \times Z) \neq (X \times Y) \times Z \qquad \text{and} \qquad X \times \{*\} \neq X \neq \{*\} \times X.$$

The symbol $\times$ above stands for the *Cartesian product* which is used in set-theory in order to combine any two sets into a new one by constructing ordered pairs out of them. We discuss Cartesian products in greater detail below and in section 4.3.1.

But for now, all this means that we cannot construct a strict monoidal category onto which mathematical structures fit. The way out of this thorny issue is to build a *non-strict* monoidal category (or simply monoidal category) by replacing *strict* equations with *isomorphic* ones. That is, we allow isomorphisms between sets such that

$$X \times (Y \times Z) \simeq (X \times Y) \times Z \qquad \text{and} \qquad X \times \{*\} \simeq X \simeq \{*\} \times X. \tag{3.5}$$

The reason we allow for isomorphisms is because, in doing so, we can construct naturality- and coherence diagrams for the additional structures we need in order to define a monoidal category onto which concrete categories fit. Moreover, these isomorphisms are not ordinary isomorphisms but so-called *natural* isomorphisms. They are an instance of the more general notion of *natural transformations* which we will define rigorously in chapter 5. A more restricted of version of this notion is in the meantime introduced in what follows.

The introduction of natural isomorphisms is a necessary step for constructing a generic monoidal category $\mathbf{C}$ for mathematical structures. We also need a monoidal product between objects and morphisms, $\otimes \in \mathbf{C}$ such that

1. for all objects $A,B \in |\mathbf{C}|$ we have

$$- \otimes - : |\mathbf{C}| \times |\mathbf{C}| \longrightarrow |\mathbf{C}| :: (A,B) \longmapsto A \otimes B, \text{ and,} \tag{3.6}$$

2. for all hom-sets $\mathbf{C}(A,B), \mathbf{C}(C,D) \in \mathbf{C}$ we have

$$- \otimes - : \mathbf{C}(A,B) \times \mathbf{C}(B,C) \longrightarrow \mathbf{C}(A \otimes C, B \otimes D) :: (f,g) \longmapsto f \otimes g, \tag{3.7}$$

By identifying mathematical structures in concrete categories with the abstract objects in monoidal categories, axiom 1 (map 3.6) enable us to compose separate mathematical structures into a single compound entity $A \otimes B$ while axiom 2 (map 3.7) allow for structure-preserving maps to map between compound entities.

Now, in order to explain the general notion of natural isomorphisms, let

$$\Gamma\left(x_1,...x_n,C_1,...,C_m\right) \qquad \text{and} \qquad \Xi\left(x_1,...x_n,C_1,...,C_m\right) \tag{3.8}$$

be two well-formed expressions constructed from

- some composition operation, $- \otimes - \in \mathbf{C}$,
- brackets,
- variables, $x_1,...,x_n \in |\mathbf{C}|$, and,
- constants, $C_1,...,C_m \in |\mathbf{C}|$.

The restricted version of a natural transformation is then a family

$$\left\{\Gamma\left(A_1,...A_n,C_1,...,C_m\right) \xrightarrow{\xi_{A_1,...,A_n}} \Xi\left(A_1,...A_n,C_1,...,C_m\right) \mid A_1,...,A_n \in |\mathbf{C}|\right\}$$

of morphisms which are such that for all objects $A_1,...,A_n,B_1,...,B_n \in |\mathbf{C}|$ and all morphisms $A_1 \xrightarrow{f_1} B_1,...,A_n \xrightarrow{f_n} B_n$ the following naturality condition commutes

$$\begin{array}{ccc} \Gamma\left(A_1,...A_n,C_1,...,C_m\right) & \xrightarrow{\xi_{A_1,...,A_n}} & \Xi\left(A_1,...A_n,C_1,...,C_m\right) \\ {\scriptstyle \Gamma\left(f_1,...f_n,1_{C_1},...,1_{C_m}\right)}\big\downarrow & & \big\downarrow{\scriptstyle \Xi\left(f_1,...f_n,1_{C_1},...,1_{C_m}\right)} \\ \Gamma\left(B_1,...B_n,C_1,...,C_m\right) & \xrightarrow[\xi_{B_1,...,B_n}]{} & \Xi\left(B_1,...B_n,C_1,...,C_m\right) \end{array} \tag{3.9}$$

Moreover, if all the morphisms $\xi_{A_1,...,A_n}$ are isomorphisms in the category-theoretical sense, then they are in fact natural isomorphisms.

We now proceed by stating all well-formed expressions and natural isomorphisms necessary in order to define a monoidal category onto which mathematical structures fit.

For the trivial expression

$$x$$

the corresponding natural isomorphism is defined from the following commuting naturality condition:

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow[id_B]{} & B \end{array} \tag{3.10}$$

For the expressions

$$x \otimes (y \otimes z) \qquad \text{and} \qquad (x \otimes y) \otimes z,$$

the corresponding natural isomorphism is defined from the following commuting naturality condition:

$$\begin{CD}
A \otimes (B \otimes C) @>{\alpha_{A,B,C}}>> (A \otimes B) \otimes C \\
@V{f \otimes (g \otimes h)}VV @VV{(f \otimes g) \otimes h}V \\
A' \otimes (B' \otimes C') @>>{\alpha_{A',B',C'}}> (A' \otimes B') \otimes C'
\end{CD} \tag{3.11}$$

The natural isomorphism α is called *associativity.* Its name refer to the fact that we use this natural isomorphism as a replacement for the *stric* associate axiom we defined for the strict monoidal category $A \otimes (B \otimes C) = (A \otimes B) \otimes C$. More explicitly, we have that composing the top morphism with the right-hand one gives the same result as composing the left-hand one with the bottom one which is all to say that it is defined in a basis independent manner.

For the expressions

$$x \quad \text{and} \quad c \otimes x \qquad x \quad \text{and} \quad x \otimes c,$$

the corresponding natural isomorphisms are defined from the following commuting naturality conditions:

$$\begin{CD}
A @>{\lambda_A}>> \mathrm{I} \otimes A \\
@V{f}VV @VV{id_{\mathrm{I}} \otimes f}V \\
B @>>{\lambda_B}> \mathrm{I} \otimes B
\end{CD}
\qquad\qquad
\begin{CD}
A @>{\rho_A}>> A \otimes \mathrm{I} \\
@V{f}VV @VV{f \otimes id_{\mathrm{I}}}V \\
B @>>{\rho_B}> B \otimes \mathrm{I}
\end{CD} \tag{3.12}$$

The natural isomorphisms λ and ρ are called *left-* and *right unit* respectively.

With the natural isomorphisms defined by the naturality conditions 3.11 and 3.12 we have now been able to reinterpret equations such as 3.5 in purely category-theoretical terms. We now only need one more natural isomorphism in order to be able to define the monoidal category. And in fact, we have already made reference to it by naturality condition 3.2.

More generally, for the expressions

$$x \otimes y \qquad \text{and} \qquad y \otimes x,$$

the corresponding natural isomorphism is defined from the following commuting naturality condition:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\sigma_{A,B}} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ C \otimes D & \xrightarrow[\sigma_{C,D}]{} & D \otimes C \end{array} \tag{3.13}$$

The natural isomorphism σ is called *symmetry.*

All of the above natural isomorphisms are *weaker* forms of axiomatization compared to the tensor structure of the strict monoidal category. Nevertheless, with these natural isomorphisms defined, we can now give the definition of the monoidal category for mathematical structures which is a crucial step in order to create physical models out of category theory.

Definition (Monoidal category). A *monoidal category* **C** contains the following information:

1. a monoidal unit $\mathrm{I} \in |\mathbf{C}|$,
2. a *bifunctor* $-\otimes-$, that is, the operation defined from 3.6 and 3.7 above, which moreover satisfies

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h) \qquad \text{and} \qquad id_A \otimes id_B = id_{A\otimes B}$$

   for all A,$B \in |\mathbf{C}|$ and all morphisms f,g,h,k of appropriate type, and,

3. three natural isomorphisms

$$\alpha = \left\{ A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C \mid A,B,C \in |\mathbf{C}| \right\},$$

$$\lambda = \left\{ A \xrightarrow{\lambda_A} \mathrm{I} \otimes A \mid A \in |\mathbf{C}| \right\} \qquad \text{and} \qquad \rho = \left\{ A \xrightarrow{\rho_A} A \otimes \mathrm{I} \mid A \in |\mathbf{C}| \right\},$$

   such that

   - the naturality conditions 3.11 and 3.12 are satisfied,
   - they satisfy a property called *coherence.*

The coherence property is that every well-formed equation built from $\circ$, $\otimes$, id, α, α^{-1},λ, λ^{-1}, ρ and ρ^{-1} must be satisfied.

Important examples of such equations are

- the *Mac Lane pentagon* equation

$$\begin{array}{ccc} & (A\otimes B)\otimes(C\otimes D) & \\ \nearrow^{\alpha_{A,B,C\otimes D}} & & \searrow^{\alpha_{A\otimes B,C,D}} \\ A\otimes(B\otimes(C\otimes D)) & & ((A\otimes B)\otimes C)\otimes D \\ \downarrow^{id_A\otimes\alpha_{B,C,D}} & & \uparrow^{\alpha_{A,B,C}\otimes id_D} \\ A\otimes((B\otimes C)\otimes D) & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & (A\otimes(B\otimes C))\otimes D \end{array} \tag{3.14}$$

for all A,B,C,$D \in |\mathbf{C}|$, and,

- the *triangle* equation

$$\begin{array}{ccc} A\otimes B & \xrightarrow{id_A\otimes\lambda_B} & A\otimes(\mathrm{I}\otimes B) \\ & \searrow^{\rho_A\otimes id_B} & \downarrow^{\alpha_{A,\mathrm{I},B}} \\ & & (A\otimes \mathrm{I})\otimes B \end{array} \tag{3.15}$$

for all A,$B \in |\mathbf{C}|$.

A monoidal category is moreover *symmetric* if in addition, there is a forth natural isomorphism

$$\sigma = \left\{ A\otimes B \xrightarrow{\sigma_{A,B}} A\otimes B \mid A,B \in |\mathbf{C}| \right\},$$

such that

- the naturality condition 3.13 is satisfied,
- it satisfy the property of coherence.

Likewise, the coherence property in this case is that every well-formed equation built from $\circ, \otimes, id, \alpha$, α^{-1},λ, λ^{-1}, ρ, , ρ^{-1}, σ and σ^{-1} must be satisfied.
Important examples of such equations include the equations 3.14 and 3.15 above and moreover

- the *triangle* equation

$$\begin{array}{ccc} A\otimes B & \xrightarrow{\sigma_{A,B}} & B\otimes A \\ & \searrow^{id_A\otimes id_B} & \downarrow^{\sigma_{B,A}} \\ & & A\otimes B \end{array} \tag{3.16}$$

for all A,$B \in |\mathbf{C}|$,

- the *triangle* equation

$$\begin{array}{ccc} A & \xrightarrow{\lambda_A} & \mathrm{I} \otimes A \\ & \searrow^{\rho_A} & \Big\downarrow \sigma_{\mathrm{I},A} \\ & & A \otimes \mathrm{I} \end{array} \qquad (3.17)$$

$$\rho_A : A \to A \otimes \mathrm{I}$$

  for all $A \in |\mathbf{C}|$, and,

- the *hexagon* equation

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\alpha_{A,B,C}} & (A \otimes B) \otimes C & \xrightarrow{\sigma_{(A\otimes B),C}} & C \otimes (A \otimes B) \\ \Big\downarrow id_A \otimes \sigma_{B,C} & & & & \Big\downarrow \alpha_{C,A,B} \\ A \otimes (C \otimes B) & \xrightarrow[\alpha_{A,C,B}]{} & (A \otimes C) \otimes B & \xrightarrow[\sigma_{A,C} \otimes id_B]{} & (C \otimes A) \otimes B \end{array} \qquad (3.18)$$

for all A,B,$C \in |\mathbf{C}|$.

For any symmetric monoidal category, equations 3.14, 3.15, 3.16, 3.17 and 3.18 must commute in order to satisfy the coherence property.

Surprisingly, it turns out that these equations are the only ones which are necessary to fulfill! Indeed, this is due to Mac Lane's *coherence theorem for symmetric monoidal categories* [20], which states:

Theorem (Coherence for symmetric monoidal categories). *Given the information for a monoidal category, α, λ, ρ and σ are coherent if and only if equations 3.14, 3.15, 3.16, 3.17 and 3.18 hold.*

This is a very important result for which the proof is highly non-trivial so we will not give a presentation of it in this project. For a comprehensive treatment, see the textbook [20]. A more modern approach to the proof can be found in [31].

As a consequence of this theorem we also have

Corollary. *Equations 3.14 and 3.15 imply*

$$\lambda_{\mathrm{I}} = \rho_{\mathrm{I}}.$$

Moreover, there exist a correspondence between monoidal categories and strict monoidal categories which further simplifies matters which is due to the following theorem which also can be found in [20]:

Theorem (Strictification). *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Monoidal equivalence determines when two monoidal categories encode the same information (see [16], [18], [20]).

What this theorem essentially tells us is that for all practical purposes, monoidal categories behave the same as strict monoidal categories. In particular, both graphical calculus and Dirac notation defined for strict monoidal categories are also valid for monoidal categories.

(Symmetric) monoidal categories were introduced independently by Bénabou and Mac Lane in 1963 [32], [33]. Early developments centred around the problem of coherence, and were resolved by Mac Lane's Coherence Theorem.

Given the structure of the monoidal category, we can now define the *braided monoidal category* as follows:

Definition. (Braided monoidal category). A *braided monoidal category* is a monoidal category $\mathbf{C}$ equipped with a family of natural isomorphisms

$$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

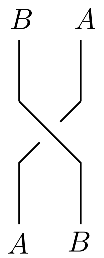
which are such that the following *hexagon* equations commute:

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B\otimes C}} & (B \otimes C) \otimes A & \xrightarrow{\alpha_{B,C,A}} & B \otimes (C \otimes A) \\ \uparrow{\scriptstyle \alpha_{A,B,C}} & & & & \uparrow{\scriptstyle id_B \otimes \sigma_{A,C}} \\ (A \otimes B) \otimes C & \xrightarrow[\sigma_{A,B}\otimes id_C]{} & (B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) \end{array} \tag{3.19}$$

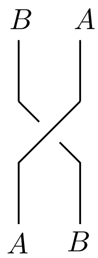
and,

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\sigma^{-1}_{B\otimes C,A}} & (B \otimes C) \otimes A & \xrightarrow{\alpha_{B,C,A}} & B \otimes (C \otimes A) \\ \uparrow{\scriptstyle \alpha_{A,B,C}} & & & & \uparrow{\scriptstyle id_B \otimes \sigma^{-1}_{A,C}} \\ (A \otimes B) \otimes C & \xrightarrow[\sigma^{-1}_{B,A}\otimes id_C]{} & (B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) \end{array} \tag{3.20}$$

We can now incorporate braiding in our graphical calculus. Indeed, the natural isomorphisms $\sigma_{A,B} : A \otimes B \longrightarrow A$ and $\sigma^{-1}_{A,B} : A \otimes B \longrightarrow B \otimes A$ are respectively depicted as:



and,



, while invertibility has the following graphical representation:

This captures part of the topological behaviour of strings.

Note in particular that, since the strings cross over each other, they are not lying on the plane but rather in three-dimensional space. Category theorists thus say that braided monoidal categories have a three-dimensional graphical calculus. In a similar fashion, concrete categories have a one-dimensional graphical calculus while monoidal categories have a two-dimensional graphical notation. Because of this, braided monoidal categories have an important connection to three-dimensional 'topological quantum field theories' (or TQFTs). We discuss TQFTs in greater detail in section 5, where we show how two-dimensional TQFTs can be defined within the framework of category theory.

Moreover, the braided monoidal category is crucial for establishing a categorical model for the behaviour of anyons as components in *toplological quantum computers*. Indeed, recall from the introduction (see section 1) that anyons inhabit a two-dimensional space and therefore, when two of them are interchanged, it is crucial to keep track of how they move around each other. That is, a clockwise rotation by π around the midpoint between two anyons is not the same as, nor even deformable to, a counterclockwise rotation. So we should describe the interchange of two anyons, $\sigma_{A,B}$, not merely as switching A with B but rather as doing so in a counterclockwise direction. Now, the choice of direction here is just a matter of convention, that is, if $\sigma_{A,B}$ is the counterclockwise direction then $\sigma_{A,B}^{-1}$ is the clockwise rotation, achieving the same interchange. Thus, in general we expect that $\sigma_{A,B} \neq \sigma_{A,B}^{-1}$. If these two were always equal, then we would have ordinary symmetries as defined in section 3.1. That is, we would not have a braided monoidal category but rather a symmetric monoidal category.

A useful mental picture is to imagine the anyons (now abstract objects composed by the monoidal product $\otimes$ in our braided category) as being lined up from left to right. The counterclockwise interchange $\sigma_{A,B}$ then amounts to moving A from the left of B to the right of B by passing A in front of B. The clockwise interchange $\sigma_{A,B}^{-1}$ would then also move A to the right of B, but it does so by passing A behind B.

In terms of moving anyons, the hexagon equation 3.19 expresses the fact that moving A past $B \otimes C$ by passing A in front of $B \otimes C$ is equivalent to first passing A in front of B and then passing A in front of C. Similarly, the hexagon equation 3.20 for the natural isomorphism $\sigma_{A,B}^{-1}$ describes the same event but passes anyons behind, instead of in front of, other anyons.

Of course, having a categorical model which describes how anyons braid is only part of the necessary structure we need in order to fully describe them. Indeed, for example, in order to categorically model the case when two anyons fuse to produce vacuum, we need some additional structure which incorporates *dual-objects*. We discuss such a structure in terms of a category known as the *compact category* in detail in section 4.2. However, although the structure of the compact category is similar to the structure one typically use for describing anyons categorically, it is not precisely the same. For a more comprehensive and detailed discussion regarding all the necessary structures needed to describe anyons we instead refer the reader to Panandagen and Paquette (2011) [6].

We now turn our attention back to the symmetric monoidal category and present some important examples of categories which fit onto this structure:

The symmetric monoidal category **Set** is the category which has associativity-, left- and right unit- and symmetry natural isomorphisms relative to the Cartesian product, with any singleton set $\{*\}$ as the monoidal unit. Explicitly, by identifying

- the bifunctor with
$$f \times f' : X \times X' \longrightarrow Y \times Y' :: \big(x,x'\big) \longmapsto \big(f(x),f'(x)\big)$$
for $f : X \to Y$ and $f' : Y \to Y'$,
- associativity with
$$\alpha_{X,Y,Z} : X \times (Y \times Z) \longrightarrow (X \times Y) \times Z :: (x,(y,z)) \longmapsto ((x,y)\,,z)$$
- left- and right unit with
$$\lambda_X : X \longrightarrow \{*\} \times X :: x \longmapsto (*, x) \qquad \text{and} \qquad \rho_X : X \longrightarrow X \times \{*\} :: x \longmapsto (x,*)$$
- symmetry with
$$\sigma_{X,Y} : X \times Y \longrightarrow Y \times X :: (x,y) \longmapsto (y,x)\,,$$

then all naturality- and coherence diagrams commutes so the category **Set** is indeed a monoidal symmetric category (see Appendix B).

In fact, **Set** actually admits two symmetric monoidal structures. The other one is obtained relative to the *disjoint union* rather than the Cartesian product. Similarly to the Cartesian product, the disjoint union can be used in order to construct a new set from any two sets. However, in contrast to the Cartesian product which produces a new set by ordering pairs of elements from any two sets, the disjoint union simply produces a set which contains all the elements from the previous sets. The disjoint union is defined by

$$X + Y := \Big\{(x,1) \mid x \in X \cup (y,2) \mid y \in Y\Big\}\,.$$

Since elements in X are ordered with 1 while elements in Y are ordered with 2, elements which are shared by both these sets will be accounted for twice within the set $X + Y$. Thus, the intersection $X \cap Y$ is empty and hence the name 'disjoint' union.

With the disjoint union defined we can alternatively consider the category **Set** to be a symmetric monoidal category by identifying

- the monoidal unit with the empty set $\emptyset$
- the bifunctor with
$$f + f' : X + X' \longrightarrow Y + Y' :: \begin{cases} (x,1) \mapsto (f(x),1) \\ (x,2) \mapsto (f'(y),1) \end{cases}$$
for $f : X \to Y$ and $f' : Y \to Y'$,

- associativity with

$$\alpha_{X,Y,Z} : X + (Y + Z) \longrightarrow (X + Y) + Z :: \begin{cases} (x{,}1) \mapsto ((x{,}1)\,{,}1) \\ ((x{,}1){,}2) \mapsto ((x{,}2){,}1) \\ ((x{,}1){,}2) \mapsto (x{,}2) \end{cases}$$

- left- and right unit with

$$\lambda_X : X \longrightarrow \emptyset + X :: x \longmapsto (x, 2) \qquad \text{and} \qquad \rho_X : X \longrightarrow X + \emptyset :: x \longmapsto (x{,}1)$$

- symmetry with

$$\sigma_{X,Y} : X + Y \longrightarrow Y + X :: (x{,}i) \longmapsto (x, 3 - i)\,.$$

Once again, with this identification all natural- and coherence conditions will be satisfied (see Appendix B).

In order to avoid confusion regarding which monoidal structure we consider we will sometimes specify the difference more explicitly so that the difference between the symmetric monoidal category $(\mathbf{Set}, \times, \{*\})$ and the symmetric monoidal category $(\mathbf{Set}, +, \emptyset)$ becomes clear.

Another example which also admits two symmetric monoidal structures is the concrete category $\mathbf{FdVect}_{\mathbb{K}}$. These are given by

1. $(\mathbf{FdVect}_{\mathbb{K}}, \otimes, \mathbb{K})$, with
   - the tensor product as monoidal product and the underlying field $\mathbb{K}$ as monoidal unit,
   - associativity identified with

     $$\alpha_{V_1,V_2,V_3} : V_1 \otimes (V_2 \otimes V_3) \longrightarrow (V_1 \otimes V_2) \otimes V_3 :: v_1 \otimes (v_2 \otimes v_3) \longmapsto (v_1 \otimes v_2) \otimes v_3$$

   - left- and right unit identified with

     $$\lambda_V : V \longrightarrow \mathbb{K} \otimes V :: v \longmapsto 1 \otimes v \qquad \text{and} \qquad \rho_V : V \longrightarrow V \otimes \mathbb{K} :: v \longmapsto v \otimes 1$$

   - symmetry identified with

     $$\sigma_{V_1,V_2} : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1 :: v_1 \otimes v_2 \longmapsto v_2 \otimes v_1.$$

2. $(\mathbf{FdVect}_{\mathbb{K}}, \oplus, \{\mathbf{0}\})$, with
   - the direct sum as monoidal product and the 0-dimensional vector space as monoidal unit,
   - associativity identified with

     $$\alpha_{V_1,V_2,V_3} : V_1 \oplus (V_2 \oplus V_3) \longrightarrow (V_1 \oplus V_2) \oplus V_3 :: v_1 \oplus (v_2 \oplus v_3) \longmapsto (v_1 \oplus v_2) \oplus v_3$$

   - left- and right unit identified with

     $$\lambda_V : V \longrightarrow \{\mathbf{0}\} \oplus V :: v \longmapsto 0 \oplus v \qquad \text{and} \qquad \rho_V : V \longrightarrow V \oplus \{\mathbf{0}\} :: v \longmapsto v \oplus 0$$

- symmetry identified with

$$\sigma_{V_1,V_2} : V_1 \oplus V_2 \longrightarrow V_2 \oplus V_1 :: v_1 \oplus v_2 \longmapsto v_2 \oplus v_1.$$

Note that for the symmetric monoidal category $(\mathbf{FdVect}_{\mathbb{K}}, \otimes, \mathbb{K})$, the inverse to λ_V is given by

$$\lambda_V^{-1} : \mathbb{K} \otimes V \longrightarrow V :: k \otimes v \mapsto k \cdot v,$$

and that the scalars are given by the field $\mathbb{K}$ itself since it is in bijective correspondence with the linear maps from $\mathbb{K}$ to itself. On the other hand, for the symmetric monoidal category $(\mathbf{FdVect}_{\mathbb{K}}, \oplus, \{\mathbf{0}\})$, there is only one scalar since there exist only one linear map from a 0-dimensional vector space to itself.

In the next chapter, we will see that this difference in the number of possible morphisms from the monoidal unit to itself has important implications regarding the monoidal product.

Now, similarly to the approach we took for strict monoidal categories, we now turn to modify the monoidal category so that a more "tailored" mathematical structure can be constructed. In the following chapter, we will make use of these more sophisticated monoidal categories in order to construct physical models. However, before we come to this point we first, gradually, introduce additional structure to the (symmetric) monodial category:

Definition (Dagger monoidal category). A *dagger monoidal category* $\mathbf{C}$ is a monoidal category equipped with an involutive identity-on-objects contravariant functor (see section 2.5)

$$\dagger : \mathbf{C}^{op} \longrightarrow \mathbf{C}$$

and in which the natural isomorphisms α, λ and ρ are all unitary.

A dagger monoidal category is moreover symmetric if, in addition, the natural isomorphism σ also is included:

Definition (Symmetric dagger monoidal category). A *symmetric dagger monoidal category* $\mathbf{C}$ is both a symmetric monoidal category and a dagger monoidal category in which the natural isomorphism σ also is unitary.

The following three examples of categories which can be considered as symmetric dagger monoidal categories are the ones that we will focus on for the rest of this project. Despite their apparent differences they are in fact structurally very similar as will become more clear as we proceed.

The first example is a familiar one, namely the concrete category **FdHilb**. In fact this concrete category admits two dagger symmetric monoidal structures given by $(\mathbf{FdHilb}, \otimes, \mathbb{C})$ and $(\mathbf{FdHilb}, \oplus, \{\mathbf{0}\})$ respectively. In both cases the adjoint is identified with the dagger functor.

The second example involves the category **Rel** which contains the following information:

1. all sets as objects,

2. all relations (multivalued functions) between sets as morphisms,
3. relation composition as morphism composition, and,
4. identity relations as unit morphisms.

We will investigate this category in greater detail in section 4.2.1. Now we only state that this category also admits two dagger symmetric monoidal structures given by $(\mathbf{Rel}, \times, \{*\})$ and $(\mathbf{Rel}, +, \emptyset)$. In both cases the *relational converse* (see section 4.2.1) is identified with the dagger functor.

As we can infer from this, the category **Rel** seen as a symmetric monoidal category has identical structure to the symmetric monoidal category **Set**. However, in contrast to the category **Set**, **Rel** moreover admits two dagger symmetric monoidal structures which is an indication that **Rel** is more closely related to the category **FdHilb** than the category **Set** is.

The third example involves the category **2Cob** which contains the following information:

1. one-dimensional closed manifolds as objects,
2. two-dimensional manifolds called *cobordisms*, which essentially are maps between different number of one-dimensional closed manifolds, as morphisms,
3. composition of cobordisms as morphism composition, and,
4. identity cobordisms as unit morphisms.

Again, we will come back to this category later in chapter 4 where give a more detailed definition of cobordisms. But what is interesting at this point is that the category **2Cob** also admits a dagger symmetric monoidal structures and is thus also structurally related to **FdHilb**. The dagger symmetric monoidal structure is given by $(\mathbf{2Cob}, +, 0)$ where 0 stands for the 'empty manifold' (see section 4.2.2). For this category, the *reversal of cobordisms* (see section 4.2.2) is identified with the dagger functor.

In the following chapter we will continue to add structure to the monoidal categories which will further show similarities in structure for the categories **FdHilb**, **Rel** and **2Cob**. It will become apparent that the category **Set** is structurally quite different from these categories. As we will show, this has to do with the monoidal product and in particular, its 'behaviour' depending on what type of underlying monoidal category it is a part of.

4

No-cloning in Categorical Quantum Mechanics

The monoidal category takes into account structural similarities within distinct mathematical structures and abstract those similarities into a single general concept. The advantage in doing so is that we then can construct modified versions of this general concept and find out which mathematical structures that fit onto those more abstract but general structures. This gives us valuable information about what exactly it is that makes some mathematical structures similar to each other. This was made explicit in the last chapter where we found that the categories **FdHilb**, **Rel** and **2Cob**, even thought at face value may appears unrelated, all could be identified as to fit onto the structure of the dagger symmetric monoidal category. That is, structurally, the categories **FdHilb**, **Rel** and **2Cob** are very similar to each other.

In this chapter we will further investigate the structural relationships between these categories and find out that these similarities are both a reflection of the monoid structure and the monoidal product. We will primarily focus on the monoidal product which, as we will explicitly show, plays an instrumental role in establishing the *no-cloning theorem of quantum information theory* purely within the framework of category theory. [34], [16], [19], [35], [36], [37].

4.1 Scalar monoid

As showed in the last chapter, the symmetric dagger monoidal category **FdHilb** incorporates enough of a rich structure in order to specify many mathematical constructs which are used within the context of quantum mechanics. In particular, the dagger functor and monoidal product can be used in order to incorporate Dirac's bra-ket formalism entirely in categorical terms. In this section we will continue this 'categorical axiomatization of quantum mechanics' by showing how scalar multiplication also can be incorporated. The key to this is to realize that scalars are in fact a general notion for all monoidal categories. Recall that scalars are obtained by considering maps of the following type:

$$s : \mathrm{I} \longrightarrow \mathrm{I},$$

that is, as an endomorphism of the monoidal unit.
Now, as we mentioned in chapter 2, for any category $\mathbf{C}$ and any $A \in |\mathbf{C}|$, the hom-set $\mathbf{C}(A, A)$ always constitutes a monoid. In particular for monoidal categories, a so-called *scalar monoid* is defined as the following hom-set

$$\mathbb{S}_{\mathbf{C}} := \mathbf{C}(\mathrm{I}, \mathrm{I})$$

, in which the composition of arrows are identified with the composition of elements.

The scalar monoid in $(\mathbf{FdHilb}, \otimes, \mathbb{C})$ is isomorphic to $\mathbb{C}$ since any linear map $s : \mathbb{C} \longrightarrow \mathbb{C}$ is, by linearity, completely determined by the image of $1 \in \mathbb{C}$, that is

$$\mathbb{S}_{(\mathbf{FdHilb}, \otimes, \mathbb{C})} \simeq \mathbb{C}.$$

In the case of $(\mathbf{Rel}, \times, \{*\})$, the corresponding scalar monoid is isomorphic to the truth values (or Booleans), *true* and *false*, usually denoted as 1 and 0 respectively. The Booleans are variables within the branch of mathematics known as *Boolean algebra*, named after George Boole which introduced and further developed this branch of mathematics, [38]. Now, since there are exactly two relations from a singleton to itself, namely the identity- and empty relation, we have that the scalar monoid in $(\mathbf{Rel}, \times, \{*\})$ is isomorphic to the Booleans. That is

$$\mathbb{S}_{(\mathbf{Rel}, \times, \{*\})} \simeq \mathbf{B}.$$

Similarly, the scalar monoid in $(\mathbf{2Cob}, +, 0)$ is isomorphic to the set of all natural numbers $\mathbb{N}$, that is

$$\mathbb{S}_{(\mathbf{2Cob}, +, 0)} \simeq \mathbb{N}.$$

What all these dagger symmetric monoidal categories have in common is that their respective scalar monoid is 'non-trivial', that is, there exist more than one possible morphism from the monoidal unit to itself. However, there are also categories for which the scalar monoid is 'trivial'.

Indeed, in the symmetric monoidal category $(\mathbf{Set}, \times, \{*\})$ the scalar monoid is isomorphic to any one-element set since there is only one function of type $\{*\} \longrightarrow \{*\}$, namely the identity function. That is

$$\mathbb{S}_{(\mathbf{Set}, \times, \{*\})} \simeq \{*\}.$$

This is also the case for $(\mathbf{FdHilb}, \oplus, \{\mathbf{0}\})$ since there is only one linear map from the 0-dimensional Hilbert space to itself, so

$$\mathbb{S}_{(\mathbf{FdHilb}, \oplus, \{\mathbf{0}\})} \simeq \{*\}.$$

As will be discussed in greater detail in the following sections, there exists a general correspondence between the number of morphisms from the monoidal unit to itself and the 'behaviour' of the monoidal product. For example, in **FdHilb** it is clear that the tensor product $\otimes$ and direct sum $\oplus$ are different monoidal products since they play very different roles within quantum theory. In particular, the joint state generated by the tensor product cannot be decomposed into the individual states while the joint state generated by the direct sum can.

Thus, we will refer to monoidal products which behaves like $\otimes$ as *non-separable*, while those which behaves like $\oplus$ will be referred to as *separable*.

A remarkable result is that the scalar monoid is always commutative. The proof, discovered by Kelly and Laplaza in 1980 [35], is given by the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{I} & \xrightarrow{\rho_{\mathrm{I}}} & \mathrm{I}\otimes\mathrm{I} & = & \mathrm{I}\otimes\mathrm{I} & \xrightarrow{\lambda_{\mathrm{I}}^{-1}} & \mathrm{I} \\
{\scriptstyle s}\uparrow & & {\scriptstyle s\otimes \mathrm{I}}\uparrow & & \downarrow{\scriptstyle id\otimes t} & & \downarrow{\scriptstyle t} \\
\mathrm{I} & \xrightarrow{\rho_{\mathrm{I}}} & \mathrm{I}\otimes\mathrm{I} & \xrightarrow{s\otimes t} & \mathrm{I}\otimes\mathrm{I} & \xrightarrow{\lambda_{\mathrm{I}}} & \mathrm{I} \\
{\scriptstyle t}\downarrow & & {\scriptstyle id\otimes t}\downarrow & & \uparrow{\scriptstyle s\otimes id} & & \uparrow{\scriptstyle s} \\
\mathrm{I} & \xrightarrow[\lambda_{\mathrm{I}}]{} & \mathrm{I}\otimes\mathrm{I} & = & \mathrm{I}\otimes\mathrm{I} & \xrightarrow[\rho_{\mathrm{I}}^{-1}]{} & \mathrm{I}
\end{array} \tag{4.1}$$

, which make use of the law of interchange, left- and right unit natural isomorphisms and $\lambda_{\mathrm{I}} = \rho_{\mathrm{I}}$.

This result has physical consequences. Indeed, due to this result we have that any monoidal category $\mathbf{C}$ always has a commutative endomorphism scalar monoid $\mathbb{S}_{\mathbf{C}}$. This implies that we cannot change quantum theory by changing the underlying complex field of the Hilbert space to another field which is not commutative, hence excluding things like 'quaternionic quantum mechanics' [39] [40].

This proof also has a second important implication. In fact, it gives us a way of describing *scalar multiplication* in a general way. That is, each scalar $s : \mathrm{I} \longrightarrow \mathrm{I}$ can be taken to induce a natural transformation

$$s_A : A \xrightarrow{\simeq} \mathrm{I} \otimes A \xrightarrow{s\otimes id_A} \mathrm{I} \otimes A \xrightarrow{\simeq} A$$

, which is established through the following commutative diagram:

$$\begin{array}{ccc}
A & \xrightarrow{s_A} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
B & \xrightarrow[s_B]{} & B
\end{array} \tag{4.2}$$

This natural transformation can now be used to define scalar multiples of a morphism $f : A \longrightarrow B$ as

$$s \bullet f := A \xrightarrow{\simeq} \mathrm{I} \otimes A \xrightarrow{s\otimes f} \mathrm{I} \otimes B \xrightarrow{\simeq} B.$$

Indeed, these scalars satisfy

$$id \bullet f = f,$$

$$s \bullet (t \bullet f) = (s \circ t) \bullet f,$$

$$(s \bullet g) \circ (t \bullet f) = (s \circ t) \bullet (g \circ f), \text{ and,}$$

$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

, which exactly generalizes the multiplicative part of the usual properties of scalar multiplication. Thus, scalars act globally in any monoidal category **C** which contain non-trivial scalars.

A fascinating observation is that being either separable or non-separable is something that depends on several factors, namely the objects, the monoidal product and the morphism-structure. Indeed, let us investigate the following table:

Table 4.1

Category	Separable	Non-separable	Other
Set	$\times$		$+$
Rel	$+$	$\times$	
FdHilb	$\oplus$	$\otimes$	
2Cob		$+$	

As we can see from this table, a monoidal product behave differently depending on which category it is a part of. For example, in **Set** we have that the Cartesian product $\times$ is separable while in **Rel** it is non-separable, while the disjoint union $+$ is separable in **Rel** while it is non-separable in **2Cob**. In **Set** it is neither, which is something that has important consequences in the branch of category theory known as *categorical logic*. Indeed, one can use the disjoint union in **Set** in order to establish the distributive law

$$A \text{ and } (B \text{ or } C) = (A \text{ and } B) \text{ or } (A \text{ and } C)$$

of classical logic [16]. For more about categorical logic we refer the reader to Samson and Gov (2001) [41].

The remaining part of this chapter will shed more light on the above table as we explain that the behaviour of a particular monoidal product is a direct consequence of the type of category it is a part of. In particular, we will show that the *no-cloning theorem of quantum information theory* is directly linked to the monoidal product $\otimes$ being non-separable in **FdHilb**.

4.2 Non-separable monoidal products

The non-separable monoidal products in table 4.1 reflects the fact that the underlying monoidal category is a so-called *compact category*. Compact categories were first introduced by Kelly in 1972 as to demonstrate a class of examples in response to the coherence problem (see section 3.4) [42]. The use of compact categories in foundations of quantum mechanics was first introduced in 2004 by Abramsky and Coecke [36]. We define the compact category as follows:

Definition (Compact category). A *compact category* **C** is a symmetric monoidal category in which every object $A \in |\mathbf{C}|$ comes with

1. an additional, *dual object* A^*,
2. a pair of morphisms

$$\mathrm{I} \xrightarrow{\eta_A} A^* \otimes A \quad \text{and} \quad A \otimes A^* \xrightarrow{\epsilon_A} \mathrm{I},$$

   respectively called *unit* and *counit*

   , and which are defined through the following commutative diagrams:

   - coherence condition on objects

$$\begin{array}{ccccc} A & \xrightarrow{\rho_A} & A \otimes \mathrm{I} & \xrightarrow{id_A \otimes \eta_A} & A \otimes (A^* \otimes A) \\ {\scriptstyle id_A}\downarrow & & & & \downarrow{\scriptstyle \alpha_{A,A^*,A}} \\ A & \xleftarrow[\lambda_A^{-1}]{} & \mathrm{I} \otimes A & \xleftarrow[\epsilon_A \otimes id_A]{} & (A \otimes A^*) \otimes A \end{array} \tag{4.3}$$

   - coherence condition on dual objects

$$\begin{array}{ccccc} A^* & \xrightarrow{\lambda_{A^*}} & \mathrm{I} \otimes A^* & \xrightarrow{\eta_A \otimes id_{A^*}} & (A^* \otimes A) \otimes A^* \\ {\scriptstyle id_{A^*}}\downarrow & & & & \downarrow{\scriptstyle \alpha^{-1}_{A^*,A,A^*}} \\ A^* & \xleftarrow[\lambda_A^{-1}]{} & \mathrm{I} \otimes A & \xleftarrow[\epsilon_A \otimes id_A]{} & (A \otimes A^*) \otimes A \end{array} \tag{4.4}$$

The unit and counit can be identified to represent the creation and annihilation of a particle-antiparticle pair respectively. However, since these involve physical entities, they are processes within some strict compact category **C** in which case the above diagrams simplify to

- coherence condition on objects

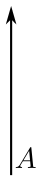
$$\begin{array}{ccc} A & & \\ {\scriptstyle id_A \otimes \eta_A}\downarrow & \searrow^{id_A} & \\ A \otimes A^* \otimes A & \xrightarrow[\epsilon_A \otimes id_A]{} & A \end{array} \tag{4.5}$$

- coherence condition on dual objects

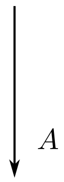
$$\begin{array}{ccc} A^* & \xrightarrow{\eta_A \otimes id_{A^*}} & A^* \otimes A \otimes A^* \\ & \searrow_{id_{A^*}} & \downarrow{\scriptstyle id_{A^*} \otimes \epsilon_A} \\ & & A^* \end{array} \tag{4.6}$$

With the above morphism-structure now defined, we can expand the graphical calculus introduced in chapter 3 so that graphical representations for the tensor structure in compact categories can be established:

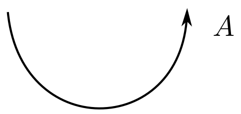
- As before, the identity morphism id_A for the object A is depicted as

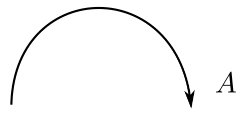


- The identity morphism id_{A^*} for the dual object of A, namely A^* is depicted as a downward arrow labelled A, that is

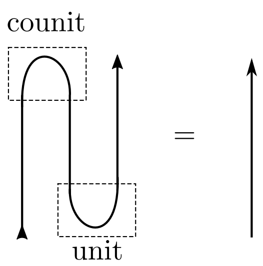


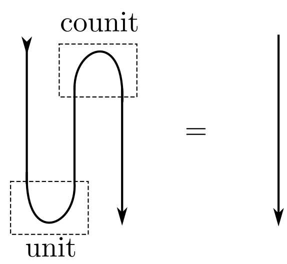
- The unit η_A and counit ϵ_A are respectively depicted as





The above commutative diagrams for the strict compact category can now be stated graphically as:



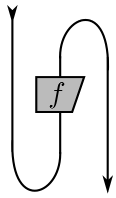


Furthermore, as we explained in chapter 3, we can also use these new *cup*- and *cap*-shaped representations for solving algebraic equations graphically. However, first we need to define the transpose to any morphism $f : A \longrightarrow B$. This is achieved by the following lemma:

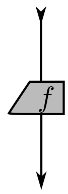
Lemma.

$$f^* := (id_{A^*} \otimes \epsilon_B) \circ (id_{A^*} \otimes f \otimes id_{B^*}) \circ (\eta_A \otimes id_{B^*}) : B^* \longrightarrow A^*, \tag{4.7}$$

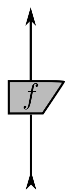
which diagrammatically is depicted as:



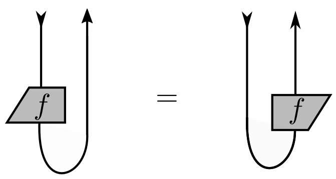
Now, in order to use these new graphical representations to solve algebraic equations in a more efficient way, we abbreviate the above picture so that it is depicted as



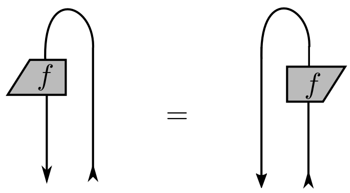
Also, recall that for morphisms of type $f : A \longrightarrow B$ the corresponding graph is given by



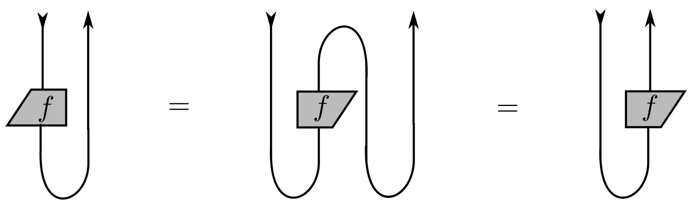
Using these graphical representations we then have



and



The proof of the first equality is given by



where we have made use of the defined transpose above. The proof for the second equality proceeds analogously.

We have thus established a way of 'sliding' boxes across wires which is rather helpful for proving algebraic equations diagrammatically as we shall see shortly. Incidentally, this allowance of 'sliding' boxes is a natural extension of the diagrammatic reasoning we established in section 3.2. Indeed, 'sliding' is permitted there as well which can been seen from the graphical representation of equation 3.1 above.

Now, for a morphism $f : A \longrightarrow B$ in a compact category, its *name* $\ulcorner f \urcorner$ is given by

$$\mathrm{I} \xrightarrow{\ulcorner f \urcorner} A^* \otimes B,$$

while its *coname* $\llcorner f \lrcorner$ is given by

$$A \otimes B^* \xrightarrow{\llcorner f \lrcorner} \mathrm{I}.$$

The name and coname are respectively defined through the following commutative diagrams:

- coherence condition for name

$$\begin{array}{ccc} A^* \otimes A & \xrightarrow{id_{A^*} \otimes f} & A^* \otimes B \\ {\scriptstyle \eta_A} \uparrow & \nearrow {\scriptstyle \ulcorner f \urcorner} & \\ \mathrm{I} & & \end{array} \tag{4.8}$$

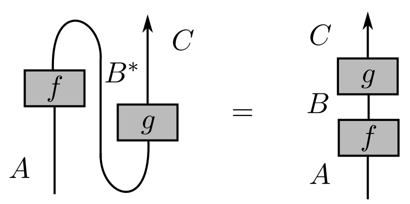
- coherence condition for coname

$$\begin{array}{ccc} & & \mathrm{I} \\ & \nearrow {\scriptstyle \llcorner f \lrcorner} & \uparrow {\scriptstyle \epsilon_B} \\ A \otimes B^* & \xrightarrow[f \otimes id_{B^*}]{} & B \otimes B^* \end{array} \tag{4.9}$$

With these additional morphisms, we can show that given $f : A \longrightarrow B$ and $g : B \longrightarrow C$ the following algebraic equation always holds

$$g \circ f = \lambda_C^{-1} \circ (\llcorner f \lrcorner \otimes id_C) \circ (id_A \otimes \ulcorner g \urcorner) \circ \rho_A.$$

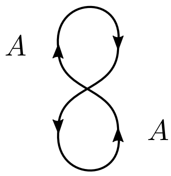
This allows us to recover composition of morphisms from their names or conames. The proof is given by this rather trivial graph:



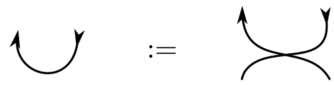
Scalars within compact categories are defined by

$$\epsilon_A \circ \sigma_{A^*,A} \circ \eta_A : \mathrm{I} \longrightarrow \mathrm{I}, \tag{4.10}$$

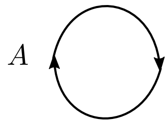
for all $A \in |\mathbf{C}|$.
The corresponding graphical representation is given by:



and if we define:



it becomes a so-called 'A-labelled circle':



An example of a category which fits onto the structure of the compact category is the category $\mathbf{FdVect}_{\mathbb{K}}$. Indeed, if we identify the linear algebraic dual space V^* to be the dual object of V, then the unit can be identified with

$$\eta_V : \mathbb{K} \longrightarrow V^* \otimes V :: 1 \longmapsto \sum_{i=1}^{n} v_i^* \otimes v_i,$$

where $\{v_i\}_{i=1}^n$ is a particular basis of V and $v_j^* \in V^*$ is the linear functional such that $v_j^*(v_i) = \delta_{i,j}$ for all $1 \leq i,j \leq n$. That is, η_V represents the coevaluation map in the compact category $\mathbf{FdVect}_{\mathbb{K}}$. Similarly, we identify the counit to be the evaluation map, that is

$$\epsilon_V : V \otimes V^* \longrightarrow \mathbb{K} :: v_i \otimes v_j^* \longmapsto v_j^*(v_i).$$

It is well known that the evaluation- and coevaluation maps can be used to characterize finite-dimensional vector spaces without referring to bases. Within the framework of category theory the same statement is made by verifying the existence of a canonical isomorphism

$$\mathbf{FdVect}_{\mathbb{K}}(V, V) \xrightarrow{\simeq} \mathbf{FdVect}_{\mathbb{K}}(\mathbb{K}, V^* \otimes V).$$

Indeed, the unit η_V is the image of id_V under this isomorphism and since id_V is independent of the basis it follows that η_V is independent as well. The argument for the counit ϵ_V proceeds analogously.

Furthermore, the V-labelled circle in $\mathbf{FdVect}_{\mathbb{K}}$ gives the dimension of a particular vector space V. Indeed, from the definition of η_V and ϵ_V together with equation 4.10 we have that a V-labelled circle amounts to

$$\sum_{ij} v_j^*(v_i) = \sum_{ij} \delta_{i,j} = \sum_{i} 1 = \dim(V).$$

Similarly, the related category **FdHilb** also fits onto the structure of the compact category. However, since **FdHilb** includes a dagger-functor it is in fact *dagger compact.* Another thing that distinguishes these categories is that the objects in **FdHilb** are self-dual. To construct the unit and counit maps for a finite-dimensional Hilbert space $\mathcal{H}$, we first pick a basis $|i\rangle$ for $\mathcal{H}$. We then construct the unit and counit respectively as follows:

1. $\eta_{\mathcal{H}} : \mathbb{C} \longrightarrow \mathcal{H} \otimes \mathcal{H} :: 1 \longmapsto \sum_i |i\rangle \otimes |i\rangle$, and,
2. $\epsilon_{\mathcal{H}} : |i\rangle \otimes |j\rangle \longmapsto \delta_{i,j}|i\rangle$.

Different bases will generate different unit and counit maps which are isomorphic to each other. We will come back to **FdHilb** in section 4.4 where we establish the no-cloning theorem of quantum information theory.

More generally we have:

Definition (Dagger compact category). A *dagger compact category* **C** is both a compact category and a dagger symmetric monoidal category such that for all $A \in |\mathbf{C}|$ we have

$$\epsilon_A = \eta_A^{\dagger} \circ \sigma_{A,A^*}. \tag{4.11}$$

4.2.1 The dagger compact category of relations

As we can infer form table 4.1, the monoidal product in both **Rel** and **FdHilb** is non-separable. In fact, they are even more similar since **Rel** is also dagger compact and has self-dual objects. Recall that **Rel** is the category which has sets as objects and relations between them as morphisms.

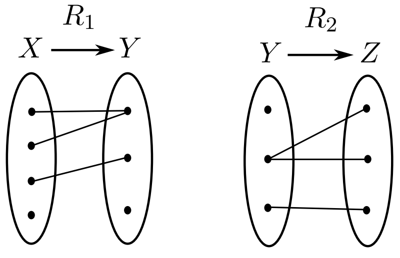
A *relation* $R : X \longrightarrow Y$ between two sets X and Y forms a subset of the set of all their order pairs, that is, $R \subseteq X \times Y$. Since the elements in R are of type $(x,y) \in R$ it is common to say that $x \in X$ *relates* to $y \in Y$, which we denote as xRy. Thus, we may express a relation as the following set:

$$R := \{(x,y) \mid xRy\},$$

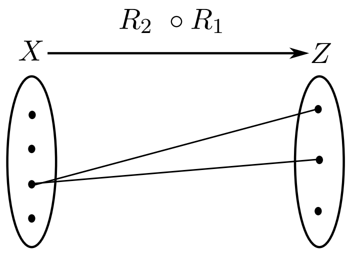
which is also referred to as the *graph* of the relation.

An example of a relation is "strictly greater than " or ">". Indeed, given two sets which only contains natural numbers, we have that 7 relates to 2 which is denoted as $7 > 2$ or $(7,2) \in > \subseteq \mathbb{N} \times \mathbb{N}$.

A more general way of thinking about a relation $R : X \longrightarrow Y$ is that it indicates the possible ways we can get from a set X to a set Y, that is,



Composition then amounts to connect up paths:



Now, in order to establish **Rel** as a dagger compact category we first have to make sure that it can be made into a symmetric monoidal category:

Definition (Symmetric monoidal category of relations). The symmetric monoidal category **Rel** contains the following information:

1. Sets are identified as objects.
2. Any singleton $\{*\}$ is identified with the monoidal unit I.
3. Relations of type $R : X \longrightarrow Y$ are identified as morphisms.
4. For $R_1 : X \longrightarrow Y$ and $R_2 : Y \longrightarrow Z$ the composite $R_2 \circ R_1 \subseteq X \times Z$ is given by

$$R_2 \circ R_1 := \{(x,z) \mid \text{there exists a } y \in Y \text{such that } xR_1y \text{ and } yR_2z\}.$$

5. The Cartesian product between sets is identified with the monoidal product between objects such that, for $R_1 : X_1 \longrightarrow Y_1$ and $R_2 : X_2 \longrightarrow Y_2$ the monoidal product is given by

$$R_1 \times R_2 := \{((x,x'),(y,y')) \mid xR_1y \text{ and } x'R_2y'\} \subseteq (X_1 \times X_2) \times (Y_1 \times Y_2).$$

6. The left- and right unit natural isomorphisms are respectively given by

$$\lambda_X := \{(x,(*,x)) \mid x \in X\} \qquad \text{and} \qquad \rho_X := \{(x,(x,*)) \mid x \in X\}.$$

7. The associativity natural isomorphism is given by

$$\alpha_{X,Y,Z} := \{((x,(y,z)),((x,y),z) \mid x \in X,\ y \in Y \text{ and } z \in Z\}.$$

8. For any X and $Y \in |\mathbf{Rel}|$, the symmetry natural isomorphism is given by

$$\sigma_{X,Y} := \{((x,y),(y,x)) \mid x \in X \text{ and } y \in Y\}.$$

With this definition we can make sure that all the coherence conditions are satisfied and thus that **Rel** indeed can be made into a symmetric monoidal category (see Appendix B).

The next step is to make **Rel** into a dagger compact category. We mentioned above that **Rel** contains self-dual objects, that is, $X^* = X$ for any $X \in |\mathbf{Rel}|$. The unit and counit are then constructed as follows:

1. $\eta_X : \{*\} \longrightarrow X \times X := \{(*, (x,x)) \mid x \in X\}$, and,
2. $\epsilon_X : X \times X \longrightarrow \{*\} := \{((x,x), *) \mid x \in X\}$.

With these identifications the coherence conditions 4.3 and 4.4 both commute thus making **Rel** into a compact category (see Appendix B).

Lastly, we need a to find a candidate to identify with the dagger

$$\dagger : \mathbf{Rel}^{op} \longrightarrow \mathbf{Rel}.$$

This can be achieved by identifying the *relational converse*. For the relation of type $R : X \longrightarrow Y$ its converse $R^{\cup} : Y \longrightarrow X$ is defined as follows:

$$R^{\cup} := \{(y,x) \mid xRy\}.$$

Indeed, by identifying the relational converse with the dagger, we can define the contravariant identity-on-objects involutive functor to be given by

1. identity-on-object property
$$\dagger : |\mathbf{Rel}^{op}| \longrightarrow |\mathbf{Rel}| :: X \longmapsto X,$$
2. assigning morphisms to their adjoints
$$\dagger : \mathbf{Rel}^{op}(X, Y) \longrightarrow \mathbf{Rel}(Y, X) :: R \longmapsto R^{\cup}.$$

Incidentally, for the relation converse the adjoint coincides with the transpose. Indeed, from the lemma 4.7 we have

$$R^* = (1_X \times \epsilon_Y) \circ (1_X \times R \times 1_Y) \circ (\eta_X \times 1_Y)$$

and since both $R^* : Y \longrightarrow X$ and $R^\dagger := R^{\cup} : Y \longrightarrow X$ we have that

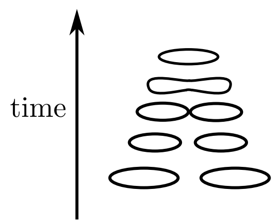
$$R^* = R^\dagger.$$

Again, with these identifications all relevant coherence conditions are satisfied (see Appendix B) so **Rel** is indeed a dagger compact category just like **FdHilb**. In contrast to these categories, **Set** does not have self-dual objects and lacks a dagger. In fact, **Set** is an entirely different type of monoidal category which is something we will get back to in section 4.4.

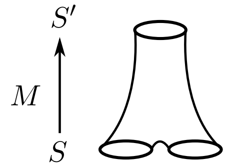
4.2.2 The dagger compact category of two-dimensional cobordisms

Just like **FdHilb** and **Rel** both include a non-separable monoidal product, so does **2Cob**, the category with one-dimensional closed manifolds as objects and two-dimensional 'cobordisms' between these as morphisms.

The *cobordisms* in **2Cob** can be thought of as describing the 'topological evolution' of one-dimensional closed manifolds through time. An example of a type of process in which corbodisms are relevant is when a pair of circles merge into one, that is,



When passing to the continuum, the same process is described by the corbodism



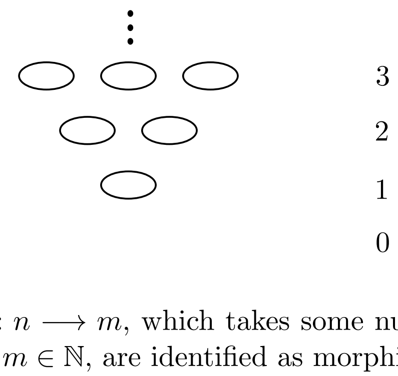
Thus, in **2Cob** the cobordisms are two-dimensional manifolds whose boundary is partitioned into two. The closed one-dimensional manifolds constitutes the domain and codomain of the cobordism. More generally, since we only are interested in the topology of the manifolds, each domain and codomain consists of some number of closed strings. Thus, similarly to both **Rel** and **FdHilb** the objects in **2Cob** are self-dual.

As we explain in greater detail in section 5, the topological nature of **2Cob** makes it very important in *topological quantum field theories* (TQFTs).

However, in this section we will instead, rather informally, show how to make **2Cob** into a dagger compact category and thus showcase the structural similarities it has with **FdHilb** and **Rel**. For a more elaborate discussion regarding the technical details we refer the reader to [11], [43], [15].

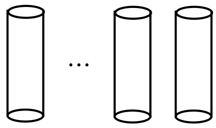
Definition (Dagger compact category of two-dimensional cobordisms). The dagger compact category **2Cob** contains the following information:

1. Finite number of closed strings are identified as objects. Hence each object can alternatively be represented by a natural number $n \in \mathbb{N}$, that is,

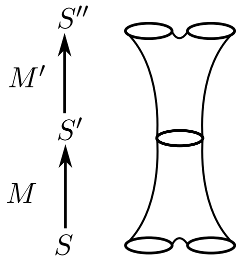


2. Cobordisms of type $M : n \longrightarrow m$, which takes some number of strings $n \in \mathbb{N}$ to some other number of strings $m \in \mathbb{N}$, are identified as morphisms. Moreover, cobordisms are defined up to homeomorphic equivalence, that is, if a cobordism can be continuously deformed into another, then they correspond to the same morphism.

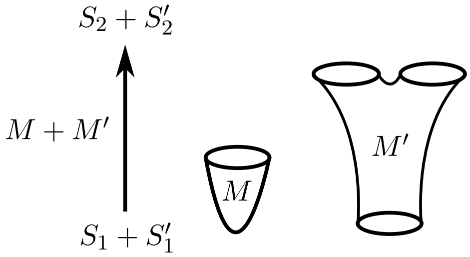
3. For any object n, the identity morphism $1_n : n \longrightarrow n$ is given by n parallel cylinders, that is,



4. Composition is identified with 'gluing' manifolds together. For example, a cobordisms of type $M' \circ M : 2 \longrightarrow 2$ is depicted as



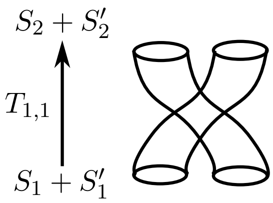
5. The disjoint union $+$ is identified with the monoidial product. For example, given cobordism $M : 0 \longrightarrow 1$ and $M' : 1 \longrightarrow 2$, then the compound cobordism $M + M' : 0 + 1 \longrightarrow 1 + 2$ is depicted as



6. The empty manifold 0 is identified with the monoidial unit.
7. The so-called *twist* is identified with symmetry. For example, the twist given by

$$T_{1,1} : 1 + 1 \longrightarrow 1 + 1,$$

is depicted as

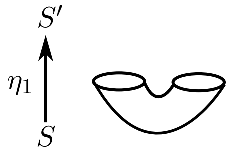


More generally, for any $n, m \in \mathbb{N}$ we have $T_{n,m} : m + n \longrightarrow n + m$.

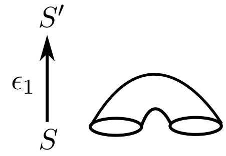
8. The unit and counit are respectively identified by the cobordisms

$$\eta_n : 0 \longrightarrow n + n \qquad \text{and} \qquad \epsilon_n : n + n \longrightarrow 0$$

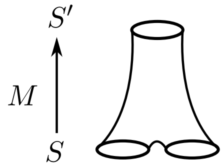
. With $n = 1$ they are respectively depicted as



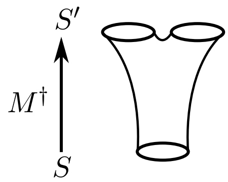
and



9. The dagger is identified by 'turning cobordisms upside down'. For example, given a cobordism of type $M : 2 \longrightarrow 1$:



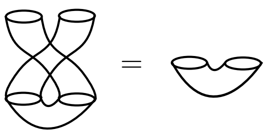
then the dagger is given by $M^\dagger : 1 \longrightarrow 2$, that is:



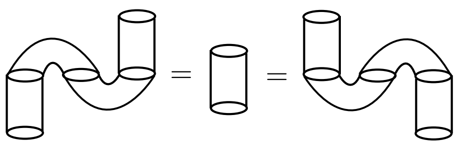
With these identifications **2Cob** fit onto the structure of the dagger compact category:

- It is a symmetric monoidal category since the dagger is compatible with the disjoint union.
- It is also dagger compact since equation 4.11 hold, which since the objects in **2Cob** are self-dual, becomes $T_{n,n} \circ \epsilon_n^\dagger = \eta_n$.

For example, with $n = 1$ we have $T_{1,1} \circ \epsilon_1^\dagger = \eta_1$ which is depicted as



Note also that we recover the same graphical representation we established at the beginning of this section by imagining that we compress the tubes in **2Cob** into strings. In particular, the composition of a unit- and counit cobordism reduces to the identity morphism just like above:



4.3 Separable monoidal products

The underlying monoidal categories for which the monoidal product is separable (see table 4.1) turn out to be incompatible with the compact structure. In fact, this incompatibility is the abstract reflection of the no-cloning theorem in quantum information theory which is the main result of this chapter.

As we can see from table 4.1, the category **Set** includes a separable monoidal product. In section 4.1 we made an important distinction between the categories $(\mathbf{Set}, \times, \{*\})$ and $(\mathbf{Rel}, \times, \{*\})$, namely that in the former we have trivial scalar structure while in the latter the scalar structure is non-trivial even though the monoidal unit, given by any one-element set $\{*\}$, is the same in both these categories. The reason for this is that the morphisms in **Rel** are relations while in **Set** they are functions, the distinction being that functions must have an output for every input while for relations this is not necessarily the case. Thus, given any set $A \in |\mathbf{Set}|$, the hom-set $\mathbf{Set}(A, \{*\})$ consist of a unique function which maps all $a \in A$ on $*$, the single element in $\{*\}$. For this reason $\{*\}$ is referred to as being *terminal* and moreover constitutes a so-called *universal property* in **Set**. We discuss more about universal properties below.

The concept of a terminal object can be dualised into that of a *initial object*. Indeed, let $\emptyset$ be the empty set, then the hom-set $\mathbf{Set}(\emptyset, A)$ again consist of a unique function, the 'empty function'. For this reason $\emptyset$ is referred to as being *initial* and again constitutes a universal property in **Set**. We generalise these properties for arbitrary categories as follows:

Definition (Terminal- and initial objects). An object $\top \in |\mathbf{C}|$ is *terminal* if there exists a unique morphism of type $A \longrightarrow \top$ for any $A \in |\mathbf{C}|$. Dually, an object $\bot \in |\mathbf{C}|$ is *initial* if there exists a unique morphism of type $\bot \longrightarrow A$ for any $A \in |\mathbf{C}|$.

That terminal and initial objects constitute universal properties is a reflection of their uniqueness in arbitrary categories:

Proposition (Universality of terminal- and initial objects). *If a category* $\boldsymbol{C}$ *has two initial- and/or terminal objects then they are isomorphic.*

Proof.
Let $\bot$ and $\bot'$ be two initial objects in an arbitrary category $\mathbf{C}$. Since $\bot$ is initial, there is a unique morphism f such that the hom-set $\mathbf{C}(\bot, \bot') = \{f\}$. By analogy, there is a unique morphism g such that the hom-set $\mathbf{C}(\bot', \bot) = \{g\}$. Now, using composition of morphisms it then follows that $g \circ f \in \mathbf{C}(\bot, \bot) = \{id_\bot\}$. Again by analogy, we also have that $f \circ g \in \mathbf{C}(\bot', \bot') = \{id_{\bot'}\}$. Therefore, $\bot \simeq \bot'$ as claimed. Analogously we also have that $\top \simeq \top'$.

■

Thus, through this proof we have established that terminal- and/or initial objects are unique up to isomorphisms. Category theorists thus often speak about "the" terminal- and/or initial object when it exists.

Another distinction between **Set** and **Rel** is that the objects in **Set** are not self-dual. These differences, namely that the monoidal unit in $(\mathbf{Set}, \times, \{*\})$ is terminal together with the fact that the objects are not self-dual, is the reason for why $(\mathbf{Rel}, \times, \{*\})$ is compact while $(\mathbf{Set}, \times, \{*\})$ is not. Indeed, this can be seen from the coname natural isomorphism.

Recall that for compact categories, a morphism of type $f : A \longrightarrow B$ comes with a coname given by $\llcorner f \lrcorner : A \otimes B^* \longrightarrow \mathrm{I}$. Now, since the monoidal unit in $(\mathbf{Set}, \times, \{*\})$ is terminal, where it true that $(\mathbf{Set}, \times, \{*\})$ was compact then the hom-set $\mathbf{Set}(A \otimes B^*, \mathrm{I})$ would have consisted of a unique morphism, namely the coname. Moreover, this would have been true for any $A \otimes B^* \in |\mathbf{Set}|$ meaning that all functions would have to be the same which is not the case. In contrast, **Rel** has relations as morphisms which means that its monoidal unit is not terminal and thus compactness can be imposed. As we will show in the following section, $(\mathbf{Set}, \times, \{*\})$ can rather be made to fit onto the structure of the so-called *Cartesian category.*

4.3.1 The Cartesian category

In section 2.3 we introduced the concept of real-world categories in which we explicitly identified physical systems as objects and processes between systems as morphisms. However, we could imagine using objects in any category to describe physical systems, and morphisms between these to describe processes. In particular, since the monoidal category **Set** includes a separable monoidal product given by the Cartesian product $\times$, it is quite useful to think about **Set** as describing *classical physics* in which a state of the joint system is just an ordered pair of states of its parts. Hence, if the first part has A_1 as its set of states, and the second part has A_2 as its set of states, the joint system has the Cartesian product $A_1 \times A_2$ as its set of states which consist of all their ordered pairs (x_1, x_2) with $x_1 \in A_1$ and $x_2 \in A_2$. We pick out a particular state in $A \in |\mathbf{Set}|$ by considering functions of the following type $f : \{*\} \longrightarrow A :: * \longmapsto x$ (see section 2.2).

Furthermore, the Cartesian product $A_1 \times A_2$ has functions called *projections* to the sets A_1 and A_2 given by

$$\pi_1 : A_1 \times A_2 \longrightarrow A_1 :: (x_1, x_2) \longmapsto x_1 \qquad \text{and} \qquad \pi_2 : A_1 \times A_2 \longrightarrow A_2 :: (x_1, x_2) \longmapsto x_2,$$

which we can imagine as maps that pick out the first or second component of any ordered pair in $A_1 \times A_2$, that is

$$\pi_1(x_1, x_2) = x_1 \qquad \text{and} \qquad \pi_2(x_1, x_2) = x_2.$$

However, these projections are not yet cast in category-theoretic terms which is something we would like them to be. That is, we would like to express the Cartesian product by means of these projections without making explicit reference to the ordered pairs. Indeed, in contrast to set theory, category theory does not recognize elements but hom-sets. Therefore, as a substitute for the above projections, we can use the following functions from an arbitrary set C to these sets:

$$\pi_1 \circ - : \mathbf{Set}(C, A_1 \times A_2) \longrightarrow \mathbf{Set}(C, A_1) :: f \longmapsto \pi_1 \circ f$$

and

$$\pi_2 \circ - : \mathbf{Set}(C, A_1 \times A_2) \longrightarrow \mathbf{Set}(C, A_2) :: f \longmapsto \pi_2 \circ f.$$

Following [16] we now combine these functions into two operations we call 'decompose'

$$dec_C^{A_1,A_2} : \mathbf{Set}(C, A_1 \times A_2) \longrightarrow \mathbf{Set}(C, A_1) \times \mathbf{Set}(C, A_2) :: f \longmapsto (\pi_1 \circ f, \pi_2 \circ f),$$

and 'recombine'

$$rec_C^{A_1,A_2} : \mathbf{Set}(C, A_1) \times \mathbf{Set}(C, A_2) \longrightarrow \mathbf{Set}(C, A_1 \times A_2) :: (f_1, f_2) \longmapsto \langle f_1, f_2 \rangle$$

where

$$\langle f_1, f_2 \rangle : C \longrightarrow A_1 \times A_2 :: c \longmapsto (f_1(c), f_2(c)).$$

For what follows, it might be useful to keep in mind that from the given operations above it follows that $\langle f_1, f_2 \rangle := f$ which has as a consequence that $(\pi_1 \circ f, \pi_2 \circ f) = (f_1, f_2)$.

With these operations we can now express the set of ordered pairs in $A_1 \times A_2$ together with their projections referring only to hom-sets. Indeed, from the above operations it follows that they are inverses of each other, that is

$$dec_C^{A_1,A_2} \circ rec_C^{A_1,A_2} = id_{\mathbf{Set}(C,A_1) \times \mathbf{Set}(C,A_2)}$$

and

$$rec_C^{A_1,A_2} \circ dec_C^{A_1,A_2} = id_{\mathbf{Set}(C,A_1 \times A_2)}.$$

Now, by defining $C := \{*\}$, we have

$$\mathbf{Set}(\{*\}, A_1 \times A_2) \underset{rec_{\{*\}}^{A_1,A_2}}{\overset{dec_{\{*\}}^{A_1,A_2}}{\rightleftarrows}} \mathbf{Set}(\{*\}, A_1) \times \mathbf{Set}(\{*\}, A_2) \tag{4.12}$$

, which corresponds to ordered pairs in $A_1 \times A_2$ being projected into elements $x \in A_1$ and $y \in A_2$ respectively. Thus, using the operations decomposition and recombination we have successfully incorporated the properties of the Cartesian products as used in set theory into the category **Set**.

Furthermore, all this generalizes to arbitrary categories. Indeed, given two objects A_1 and A_2 in some category $\mathbf{C}$, we define their *Cartesian product* (or simply *product*) as follows

Definition (Product). A *product* of two objects A_1 and $A_2 \in |\mathbf{C}|$ consist of the following information:

1. Another object $A_1 \times A_2 \in |\mathbf{C}|$ which comes with two morphisms called *projections*, namely
2. $\pi_1 : A_1 \times A_2 \longrightarrow A_1$ and,
3. $\pi_2 : A_2 \times A_2 \longrightarrow A_2$

   , and which is such that for all $C \in |\mathbf{C}|$ the mapping

   $$(\pi_1 \circ -, \pi_2 \circ -) : \mathbf{C}(C, A_1 \times A_2) \longrightarrow \mathbf{C}(C, A_1) \times \mathbf{C}(C, A_2)$$

   admits an inverse $\langle -, - \rangle_{C,A_1,A_2}$.

The product may not exist, and it may not be unique. However, if it does exist it is unique up to a canonical isomorphism:

Proposition (Product uniqueness up to isomorphism). *If a pair of objects admits two distinct products then they are isomorphic.*

Proof. Suppose that A_1 and $A_2 \in |\mathbf{C}|$ have two products $A_1 \times A_2$ and $A_1 \boxtimes A_2$ with the following respective projections

$$\pi_i : A_1 \times A_2 \longrightarrow A_i \qquad \text{and} \qquad \pi'_j : A_1 \boxtimes A_2 \longrightarrow A_j,$$

where i,j takes the values 1 or 2.
These projections are contained in the following hom-sets:

$$(\pi'_1, \pi'_2) \in \mathbf{C}(A_1 \boxtimes A_2, A_1) \times \mathbf{C}(A_1 \boxtimes A_2, A_2)$$

and

$$(\pi_1, \pi_2) \in \mathbf{C}(A_1 \times A_2, A_1) \times \mathbf{C}(A_1 \times A_2, A_2),$$

both of which we can 'recombine' using the inverse $\langle -, - \rangle_{C,A_1,A_2}$ from the definition of product above. That is, applying the respective inverses of

$$(\pi_1 \circ -, \pi_2 \circ -) \qquad \text{and} \qquad (\pi'_1 \circ -, \pi'_2 \circ -)$$

to these hom-sets yields

$$Rec^{A_1,A_2}_{A_1 \boxtimes A_2} : \mathbf{C}(A_1 \boxtimes A_2, A_1) \times \mathbf{C}(A_1 \boxtimes A_2, A_2) \longrightarrow \mathbf{C}(A_1 \boxtimes A_2, A_1 \times A_2) :: (\pi'_1, \pi'_2) \longmapsto f$$

and

$$Rec^{A_1,A_2}_{A_1 \times A_2} : \mathbf{C}(A_1 \times A_2, A_1) \times \mathbf{C}(A_1 \times A_2, A_2) \longrightarrow \mathbf{C}(A_1 \times A_2, A_1 \boxtimes A_2) :: (\pi_1, \pi_2) \longmapsto g,$$

where $f := \langle \pi_1 \circ \pi'_1, \pi_2 \circ \pi'_2 \rangle$ and $g := \langle \pi'_1 \circ \pi_1, \pi'_2 \circ \pi_2 \rangle$.
We thus have

$$\pi'_1 = \pi_1 \circ f, \quad \pi'_2 = \pi_2 \circ f, \quad \pi_1 = \pi'_1 \circ g \quad \text{and} \quad \pi_2 = \pi'_2 \circ g,$$

from which it follows that

$$(\pi'_1 \circ id_{A_1 \boxtimes A_2}, \pi'_2 \circ id_{A_1 \boxtimes A_2}) = (\pi_1 \circ f, \pi_2 \circ g) = (\pi'_1 \circ g \circ f, \pi'_1 \circ g \circ f),$$

which is an morphism in $\mathbf{C}(A_1 \boxtimes A_2, A_1) \times \mathbf{C}(A_1 \boxtimes A_2, A_2)$.

Similarly to above, 'recombining' this hom-set by applying the inverse of $(\pi'_1 \circ -, \pi'_2 \circ -)$ now gives $id_{A_1 \boxtimes A_2} = g \circ f$. By an analogous argument we also have $id_{A_1 \times A_2} = f \circ g$ which is to say that f is an isomorphism between the two objects $A_1 \times A_2$ and $A_1 \boxtimes A_2$ with g as its inverse. ■

Thus, through this proof we have established that products are unique up to isomorphisms. Similarly to the case where terminal- and/or initial objects exist, category theorists also here feel free to speak about "the" product when it exist.

Incidentally, a terminal object can be thought of in terms of products. We say a category has *binary products* if every pair of objects has a product. More generally we may also speak about n-ary products. For example, for $n = 3$, we have that every triplet of objects has a product. Moreover, if a category contains binary products then n-ary products for $n \geq 1$ exist as well, since we can construct these as iterated binary products. Now, the case when $n = 1$ is trivial since a product of one object is the object itself. On the other hand, for $n = 0$ we obtain the terminal object, that is, the object for which, for any $X \in |\mathbf{C}|$, the morphism of type $X \longrightarrow \top$ is unique. Thus, if a category has a terminal object and binary products, it has n-ary products for all n, to which we say that the category contains *finite products.*

A more conventional way of defining products, which is more common in the literature, proceeds as follows:

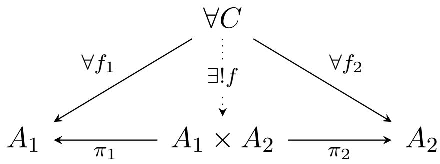
Definition. (Product). A *product* of two objects A_1 and $A_2 \in |\mathbf{C}|$ consist of the following information:

1. Another object $A_1 \times A_2 \in |\mathbf{C}|$ which comes with two morphisms called *projections*, namely
2. $\pi_1 : A_1 \times A_2 \longrightarrow A_1$ and,
3. $\pi_2 : A_2 \times A_2 \longrightarrow A_2$

   , and which is such that for all $C \in |\mathbf{C}|$, and any pair of morphisms $f_1 : C \longrightarrow A_1$ and $f_2 : C \longrightarrow A_2$ in $\mathbf{C}$, there exists a unique morphism $f : C \longrightarrow A_1 \times A_2$ such that

$$f_1 = \pi_1 \circ f \qquad \text{and} \qquad f_2 = \pi_2 \circ f.$$

As we mentioned above, the fact that this definition contains uniqueness in form of the morphism $f : C \longrightarrow A_1 \times A_2$ means that this morphism constitutes a universal property. We can summarise the definition of products together with this universality property by the following commutative diagram:



(4.13)

Universal properties are abundant throughout mathematics and are of great interest to category theorists. The abstract nature of category theory helps to identify and generalise these properties such that they more easily can be recognised within different areas of mathematics. For more details regarding universal properties and the above definition we refer the reader to [44], [20], [18].

Having discussed products we are now in position to define the category onto which $(\mathbf{Set}, \times, \{*\})$ fit:

Definition (The Cartesian category). A category $\mathbf{C}$ is *Cartesian* if it admits finite products, that is, if $\mathbf{C}$ contains a terminal object $\top \in |\mathbf{C}|$ and for any pair of objects A,$B \in |\mathbf{C}|$ there exist a product.

Proposition (Cartesian categories admits symmetric monoidal structure). *Given a Cartesian category* $\boldsymbol{C}$*, each choice of a product for each pair of objects always defines a symmetric monoidal structure on* $\boldsymbol{C}$ *with* $A \otimes B := A \times B$*, and where the terminal object* $\top$ *is identified with the monoidal unit* I.

We present a proof of this proposition in Appendix C.

In the following section we will continue the discussion regarding **Set** being a model of classical physics. In fact, all Cartesian categories are such models. This will become more concrete as we show that objects within Cartesian categories can be copied. In contrast, we use **FdHilb** to model quantum physics which, as wee saw in section 4.2, fit onto the structure of the compact category. As we will show, this has as a consequence that 'quantum states' cannot be copied.

4.4 Copy- and delete ability

The structure captured by the definition of the Cartesian category above turns out to be a good model for much of our intuition about joint systems in classical physics. For simpler versions of classical physics, the elements within some set can be identified with the states while for more elaborate versions of classical physics we can instead identify states by objects in some more sophisticated Cartesian category, such as the category of topological spaces. In both these categories finite products are used to describe the state of a joint system.

Indeed, similarly to real-world categories, where the objects are explicitly identified with physical systems, any Cartesian category also contains objects which we can use to describe physical systems in which the states are identified by morphisms of type $f : \top \longrightarrow A$. For example, if our Cartesian category is **Set**, this type of morphism picks out an element of the set A (see section 2.2). If instead the Cartesian category is that of topological spaces, then a morphism of this type corresponds to picking out a point in the topological space A. A morphism of type $f : \top \longrightarrow A$ is therefore often called an *element* of the object A.

Again, similarly to real-world categories, where the morphisms are explicitly identified with processes, morphisms in Cartesian categories of type $g : A \longrightarrow B$ also correspond to processes in which the states of physical system A are mapped over to the states of physical system B. Indeed, for any state in A, a corresponding morphism of type $f : \top \longrightarrow A$ can be composed with a morphism of appropriate type $g : A \longrightarrow B$ in order to get a state in B, that is, $g \circ f : \top \longrightarrow B$.

Furthermore, given two physical systems that are described by the objects A and B respectively, we have that the joint system built from these is described by the object $A \times B$. We can then imagine that the projections $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$ correspond to processes which takes a state of the joint system and discards all information concerning one of the composed systems, leaving information about the other system.

However, in order to model quantum theory, we must change the above model drastically! As we saw in section 4.2, the category which involves finite-dimensional Hilbert spaces, **FdHilb**, is compact rather than Cartesian. The states of a quantum system can still be thought of as forming a set. However, we do not use the product of these sets to represent the set of the joint system. Indeed, using **FdHilb**, we describe states of a quantum system as unit vectors in a Hilbert space, modulo phases. So the Hilbert space of a joint system is rather given by the tensor product, that is, given two quantum systems with Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the joint system is given by $\mathcal{H} \otimes \mathcal{K}$. This tensor product is not Cartesian in the sense defined above, since given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ there are no linear maps of types $\pi_1 : \mathcal{H} \otimes \mathcal{K} \longrightarrow \mathcal{H}$ and $\pi_2 : \mathcal{H} \otimes \mathcal{K} \longrightarrow \mathcal{K}$ with the required properties. This means that from a (pure) state of a joint quantum system we cannot extract (pure) states of its parts. Incidentally, this is the key to Bell's '*failure of local realism*' [45]. Indeed, one can derive Bell's inequality from the assumption that pure states of a joint system determines pure states of its parts [46], so violations of Bell's inequality should be seen as an indication that this assumption fails. Furthermore, the fact that the tensor product $\otimes$ in **FdHilb** is not Cartesian, means that '*we cannot clone a quantum state*', [47].

Indeed, for any monoidal category $\mathbf{C}$ for which an object A admits a product $A \times A$, there exist a unique natural transformation

$$\Delta = \left\{ A \xrightarrow{\Delta_A} A \times A \;\middle|\; A \in |\mathbf{C}| \right\},$$

called the *diagonal*[1] of A. For any object $A \in |\mathbf{C}|$, the corresponding naturality condition is given by

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow{\scriptstyle \Delta_A} & & \Big\downarrow{\scriptstyle \Delta_B} \\ A \otimes A & \xrightarrow[f \otimes f]{} & B \otimes B \end{array} \tag{4.14}$$

That is, 'performing an operation f on a system A and then copying it', is the same as 'copying system A and then performing the operation f on each copy'.

Thus, given an object A with the product being $A \times A$, the morphism $\Delta : A \longrightarrow A \times A$ is such that $\pi_1 \circ \Delta = id_A$ and $\pi_2 \circ \Delta = id_A$. That is, the role of the diagonal morphism is to duplicate information, just as the projections discards information. In relation to physics, the equations $\pi_1 \circ \Delta = id_A$ and $\pi_2 \circ \Delta = id_A$ says that if we duplicate a state in A and then discard one of the resulting copies, we end up with a copy identical to the original.

For example, the Cartesian category **Set** has

$$\{\Delta_X : X \longrightarrow X \times X :: x \longmapsto (x, x) \;\mid\; X \in |\mathbf{Set}|\}$$

as a diagonal since the following diagram commutes for any $X \in |\mathbf{Set}|$:

$$\begin{array}{ccc} X & \xrightarrow{x \mapsto f(x)} & Y \\ \Big\downarrow{\scriptstyle x \mapsto (x,x)} & & \Big\downarrow{\scriptstyle f(x) \mapsto (f(x), f(x))} \\ X \times X & \xrightarrow[(x,x) \mapsto (f(x), f(x))]{} & Y \times Y \end{array} \tag{4.15}$$

However, in the compact category **FdHilb**, since the tensor product is not a product in the category-theoretical sense, it makes no sense to speak of a diagonal morphism $\Delta : \mathcal{H} \longrightarrow$

[1] The name, 'diagonal' of A, refers to the fact that in **Set**, it is the map given by $\Delta(a) = (a,a)$ for all $a \in A$ whose graph is a diagonal line when A is the set of real numbers.

$\mathcal{H} \otimes \mathcal{H}$. More explicitly, there is no basis-independent way to choose a linear map from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$ other than the zero map. Thus, we cannot duplicate information in quantum theory.

Indeed, let us show this more explicitly by specifying some basis $\{|i\rangle\}_i$ for each Hilbert space $\mathcal{H} \in |\mathbf{FdHilb}|$. We can then consider

$$\{\Delta_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H} :: |i\rangle \longmapsto |i\rangle \otimes |i\rangle \ \mid \ \mathcal{H} \in |\mathbf{FdHilb}|\} \, .$$

But now the corresponding diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{1 \mapsto |0\rangle + |1\rangle} & \mathbb{C} \oplus \mathbb{C} \\ {\scriptstyle 1 \mapsto 1 \otimes 1} \big\downarrow & & \big\downarrow {\scriptstyle \begin{array}{l} |0\rangle \mapsto |0\rangle \otimes |0\rangle \\ |1\rangle \mapsto |1\rangle \otimes |1\rangle \end{array}} \\ \mathbb{C} \simeq \mathbb{C} \otimes \mathbb{C} & \xrightarrow[1 \otimes 1 \mapsto (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)]{} & (\mathbb{C} \oplus \mathbb{C}) \otimes (\mathbb{C} \oplus \mathbb{C}) \end{array} \tag{4.16}$$

fails to commute, since via one path we obtain the (unnormalized) *Bell-state*

$$1 \longmapsto |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle$$

while via the other path we obtain an (unnormalized) *disentangled state*

$$1 \longmapsto (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle).$$

Thus, the fact that **FdHilb** is compact has a as a consequence that no natural diagonal can be defined which in turn implies that that we cannot copy quantum states. Indeed, this can be inferred from diagram 4.16 which therefore constitutes a categorical proof for the no-cloning theorem in quantum information theory. Historically, the no-cloning theorem was proved in 1982 independently by Wootters and Zurek, and Dieks [48], [49]. The categorical version we presented here is due to Abramsky in 2010 [50]. This proof, given by diagram 4.16, is both short and elegant but moreover explains the fundamental mathematical reason for why we cannot clone objects in **FdHilb**, namely the lack of products in the technical sense given by diagram 4.13.

An interesting point can be made here. As we explained at the beginning of this section, the Cartesian category **Set** captures much of our intuition about joint systems in classical physics. This is because the concept of a set naturally formalizes some of our intuitions about countable objects in nature such as, for example, a pile of pebbles. It may thus seem amazing that the mathematics based on set theory can successfully describe quantum mechanics as well. However, this should not really come as a surprise since set theory is flexible enough to accommodate, in the language of set theory, any sort of effective algorithm for making predictions. However, the mere fact that we can use set theory as a framework for studying quantum theory does not necessarily imply that it is the most enlightening approach. Indeed, quantum phenomenon are famously counter-intuitive which suggests that not only set theory but even classical logic is not ideal for understanding quantum systems. To the category theorist, this raises the possibility that quantum theory might make more sense when viewed, not from **Set**, but within some other category, for example, **FdHilb**. In this thesis, we have stated precisely how intuitions taken from **Set** fail in **FdHilb**, namely that unlike **Set**, **Fd-Hilb** is dagger-compact where the tensor product is non-Cartesian. A more concrete and elaborate discussion regarding this idea can be found in [51].

Now, since the categories **Rel** and **2Cob** are compact as well, a similar argument can be made here. For **Rel**, given that every function is also a relation, let us consider the diagonal to be the same as in **Set**, that is, the diagonal in **Set** expressed in relational notation:

$$\Delta_X := \{(x,(x,x)) \mid x \in X\} \subseteq X \times (X \times X).$$

However, since **Rel** is compact, the corresponding diagram

$$\begin{array}{ccc} \{*\} & \xrightarrow{\{(*,0),(*,1)\}} & \{0,1\} \\ {\scriptstyle \{(*,(*,*))\}}\downarrow & & \downarrow{\scriptstyle \{(0,(0,0)),(1,(1,1))\}} \\ \{(*,*)\} = \{*\}\times\{*\} & \xrightarrow[\{(*,0),(*,1)\}\times\{(*,0),(*,1)\}]{} & \{0,1\}\times\{0,1\} \end{array} \tag{4.17}$$

fails to commute. Indeed, for the upper right path we have

$$\{*\} \xrightarrow{R} \{0{,}1\} \xrightarrow{\Delta_{\{0,1\}}} \{0{,}1\} \times \{0{,}1\},$$

that is,

$$\Delta_{\{0,1\}} \circ R = \Big\{(*,(0'',0''),(*,(1'',1'')) \;\Big|\; \exists\,(0',1') \in \{0,1\} \text{ s.t} \\ *\,R(0',1') \text{ and } (0',1')\Delta\big((0',(0'',0'')),(1',(1'',1''))\big)\Big\}, \tag{4.18}$$

which by the definition of R and Δ simplifies to

$$\{(*,(0,0)),(*,(1,1))\} = \{*\} \times \{(0,0),(1,1)\}.$$

For the lower left path we have

$$\{*\} \xrightarrow{\Delta\{*\}} \{*\} \times \{*\} \xrightarrow{R_1 \times R_2} \{*\} \times \{*\},$$

where $R_1 : \{*\} \longrightarrow \{0,1\}$ and $R_2 : \{*\} \longrightarrow \{0,1\}$ so that $R_1 \times R_2 \subseteq \{*\} \times \{*\} \longrightarrow \{0,1\} \times \{0,1\}$. Unfolding the composition $R_1 \times R_2 \circ \Delta_{\{*\}}$ we have

$$R_1 \times R_2 \circ \Delta_{\{*\}} = \Big\{(*,(0'',0'')),(*,(0'',1'')),(*,(1'',0'')),(*,(1'',1'')) \;\Big|\; \exists\,(*',*') \in \{*\}\times\{*\} \text{ s.t} \\ *\,\Delta(*',*') \text{ and } (*',*')R_1 \times R_2\big(((0'',0''),(0'',1'')),((1'',0''),(1'',1''))\big)\Big\}, \tag{4.19}$$

which by the definition of $R_1 \times R_2$ and Δ simplifies to

$$\{(*,(0,0)),(*,(0,1)),(*,(1,0)),(*,(1,1))\} = \{*\} \times (\{0,1\} \times \{0,1\}).$$

Thus, since diagram 4.17 fails to commute, the objects in **Rel** cannot be copied.

For **2Cob**, let us consider the following diagonal:

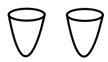
$$\{\Delta_n : n \longrightarrow n + n \mid n \in \mathbb{N}\}.$$

The corresponding diagram is given by

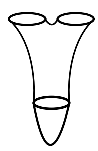
$$\begin{array}{ccc} 0 & \xrightarrow{\Delta_0} & 0+0 \\ {\scriptstyle M}\Big\downarrow & & \Big\downarrow{\scriptstyle M+M} \\ 1 & \xrightarrow[\Delta_1]{} & 1+1 \end{array} \tag{4.20}$$

Recall from section 4.2.2 that the morphism $M : 0 \longrightarrow 1$ is depicted as

Now, diagram 4.20 fails to commute since taking the upper right path yields



while the lower left path gives



Thus, since the structurally related categories **FdHilb**, **Rel** and **2Cob** are all compact, there does not exist any uniform[2] copying operation for any of them. In contrast, since **Set** is Cartesian, a uniform copying operation does exist. Indeed, for arbitrary Cartesian categories we have the following result:

Proposition. *All Cartesian categories admit a uniform copying operation.*

We present a proof of this proposition in Appendix D.

Before we conclude this chapter, let us first compare the categories **Rel** and **Set** more closely. Superficially these categories seem to be very similar as both **Rel** and **Set** have sets as objects and the Cartesian product as monoidal product. However, through the results given from diagrams 4.15 and 4.17 we have that the diagonal in **Set** fails to be a diagonal in **Rel**. This implies that the Cartesian product between sets in **Rel** does not entail a product in the sense of diagram 4.13 above.

To show this explicitly, let us use diagram 4.13 for **Rel**:

[2] To say that we have 'uniform' copying means that it does not matter if we copy something right away, or first process it for a while and then copying the result.

$$\begin{array}{ccccc} & & \{*\} & & \\ & \swarrow^{\emptyset} & \downarrow{\scriptstyle \exists! f} & \searrow^{id_{\{*\}}} & \\ \{*\} & \xleftarrow[\pi_1]{} & \{*\} \times \{*\} & \xrightarrow[\pi_2]{} & \{*\} \end{array} \tag{4.21}$$

where $\emptyset$ stands for the empty relation. That is, a relation in some set A is called an empty relation if the relation is such that no elements in A can be related, or more accurately, the empty relation $\emptyset$ on a non-empty set A is vacuously symmetric and transitive but not reflexive. Thus, given a non-empty set A, and since the empty relation by definition must be a subset $\emptyset \subseteq A \times A$, we have that the the empty relation amounts to the empty set $\{\,\}$.

Now, since $\{*\} \times \{*\} = \{(*, *)\}$ is a singleton, in order to make diagram 4.21 commute we are left with two possible choices for π_1 and π_2, namely the empty relation and the singleton relation $\{((*, *), *)\} \subseteq \{(*, *)\} \times \{*\}$. Similarly, the unique relation f can also only be one of two possible candidates, namely the empty relation and the singleton relation $\{(*, (*, *))\} \subseteq \{*\} \times \{(*, *)\}$. So, since $\pi_2 \circ f = id_{\{*\}}$ we must have that $f = \{(*, (*, *))\}$ and $\pi_2 = \{((*, *), *)\}$. Thus, π_1 is the empty relation while π_2 is the singleton relation. However, we can also use diagram 4.13 for **Rel** to obtain:

$$\begin{array}{ccccc} & & \{*\} & & \\ & \swarrow^{id_{\{*\}}} & \downarrow{\scriptstyle \exists! f} & \searrow^{\emptyset} & \\ \{*\} & \xleftarrow[\pi_1]{} & \{*\} \times \{*\} & \xrightarrow[\pi_2]{} & \{*\} \end{array} \tag{4.22}$$

We then have that π_2 must be the empty relation while π_1 must be the singleton relation, so we end up with a contradiction. This contradiction is linked to the fact that relations, in contrast to functions, need not be *total*. That is, for relations, each argument does not necessarily need to be assigned a value. Another distinction between relations and functions is that relations can be multivalued while functions are always single-valued, that is, each element of the function's domain maps to a single, well-defined element of its range. We made use of the multivalued property of relations when we showed that the diagonal in **Set** was not a diagonal in **Rel**. This can be seen by considering the the upper path in diagram 4.17 where the relation is given by $\{(*, 0), (*, 1)\} \subseteq \{*\} \times \{0, 1\}$. Hence, the fact that relations can be multivalued obstructs the existence of a diagonal in **Rel**, while the fact that relations need not be total obstructs the existence of faithful projections in **Rel**, which implies that the Cartesian product $\times$ in **Rel** does not constitute a product in the technical sense it does in **Set**.

With this comparison we conclude this chapter. We have developed two fundamentally different structures, namely the dagger-compact structure onto which the categories **FdHilb**, **Rel** and **2Cob** fit, and the Cartesian structure onto which the category **Set** fit. We also made two comparisons, one between the categories **Set** and **Rel** and one between the categories **Set** and **FdHilb**. From the former comparison we concluded that **Rel** and **Set** are structurally quite different from each other, even though they naively may seem very similar. From the latter comparison we concluded that the ability to copy objects was inherently linked to Cartesian categories in which any pair of objects admits a product. Therefore, the fact that **FdHilb** is dagger-compact where the tensor product is non-Cartesian, led us to the central result of this chapter, namely the categorical version of the no-cloning theorem in quantum information theory.

What is truly remarkable is that these ways in which **FdHilb** differs from **Set** are precisely the ways it resembles **2Cob**, the category of two-dimensional cobordisms going between one-dimensional manifolds. The similarities in structure between categories **FdHilb** and **2Cob** has led to the proposition that category theory may have a central role in the goal of unifying quantum mechanics with general relativity! This point is made by John Baez in [43] where he points out that the one-dimensional manifolds represents space while the two-dimensional cobordisms represents spacetime. Thus, from the categorical point of view, since **FdHilb** and **2Cob** resembles eachother far more than either resembles **Set**, a linear map between Hilbert spaces acts more like a spacetime than a function. In addition to these categorical ideas there are also clues towards a unification of general relativity and quantum mechanics from the work on so-called 'topological quantum field theories' (or TQFTs) which we will return to in the following chapter. TQFT's possess certain features one expects from a theory of quantum gravity [12], [13], however, this is not to say that quantum gravity is, or will be, a TQFT. TQFTs are mostly used as "baby-models" in which calculations merely gives insight into a more full-fledged theory of quantum gravity. In the following and last chapter of this thesis we will show how the mathematical definition of a TQFT can be expressed categorically. We will not delve further into the very technical connection between TQFTs and a potential theory of quantum gravity within the framework of category theory. To develop all relevant notions would take us far beyond the scope of this more pedagogical introduction to category theory. For more regarding these ideas we instead refer the reader to the following paper by John Baez and James Dolan, [52]. A more pedagogical introduction to the idea can be found in [11].

5

A Categorical Construction of Topological Quantum Field Theories

Topological quantum field theories (TQFTs) are quantum field theories that compute topological invariants. They are primarily used in condensed matter physics to describe, for instance, the fractional quantum Hall effect. TQFTs have also become a primary study for quantum gravity in which they describe an 'imaginary' world in which everywhere looks like everywhere else. More accurately, they are *background-free quantum theories with no local degrees of freedom* which seem to propose an important analogy between the mathematics of spacetime and the mathematics of quantum theory [12], [13]. In this chapter we will show how the mathematical definition of a generic TQFT can be expressed purely in categorical terms. We will consider Atiyah's definition of a generic TQFT, [53], and as we will see, the categorical restatement of Atiyah's definition relies crucially on the categories $\mathbf{FdVect}_{\mathbb{K}}$ and **nCob**. However, before we get to this point we must first develop some relevant notions of category theory we have not yet encountered.

The bulk of this section is heavily inspired by the last chapter in Bob Coecke's and Éric Oliver Paquette's excellent review paper from 2009 [16] to which we refer the reader for a more detailed discussion. Here we have merely rearranged some sections which can be found there but also added some information from the following papers [11], [43], [54].

5.1 Internal classical structures

As we saw in the last section, all Cartesian categories admits a uniform copying operation. In fact, a stronger statement can be made:

Proposition. *All Cartesian categories admits a uniform copying operation and a uniform deleting operation.*

In order to verify this statement we first need to introduce the notion of an *internal comonoid*:

Definition (internal comonoid). Given a monoidal category $(\mathbf{C}, \otimes, \mathrm{I})$, an *internal comonoid* is an object $C \in |\mathbf{C}|$ together with a pair of morphisms

$$C \otimes C \xleftarrow{\delta} C \xrightarrow{\epsilon} \mathrm{I},$$

where δ is the *comultiplication* and ϵ is the *comultiplicative unit*, which are defined through the commutation of the following two diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\quad\delta\quad} & C \otimes C \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle id_C \otimes \delta} \\ C \otimes C & \xrightarrow[\delta \otimes id_C]{} & C \otimes C \otimes C \end{array} \tag{5.1}$$

and,

$$\begin{array}{ccccc} & \swarrow^{\simeq} & C & \searrow^{\simeq} & \\ & & {\scriptstyle\delta}\downarrow & & \\ \mathrm{I} \otimes C & \xleftarrow[\epsilon \otimes id_C]{} & C \otimes C & \xrightarrow[id_C \otimes \epsilon]{} & C \otimes \mathrm{I} \end{array} \tag{5.2}$$

Regarding Cartesian categories, the morphisms δ and ϵ corresponds to copying and deleting respectively. In fact, since any Cartesian category can be endowed with symmetric monoidal structure (see section 4.3.1 and appendix D) the internal comonoid is commutative meaning that for any object $A \in |\mathbf{C}|$, the following equation is satisfied:

$$\sigma_{A,A} \circ \delta = \delta.$$

Thus, an alternative definition of Cartesian categories can be stated in terms of the existence of a uniform copying operation (the diagonal Δ introduced in section 4.3.1) and a corresponding uniform deleting operation

$$\mathcal{E} = \left\{ A \xrightarrow{\mathcal{E}_A} \mathrm{I} \;\middle|\; A \in |\mathbf{C}| \right\},$$

for which the naturality constraint is given by the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle\mathcal{E}_A}\downarrow & \swarrow_{\mathcal{E}_B} & \\ \mathrm{I} & & \end{array} \tag{5.3}$$

That is, for each object A in any Cartesian category, the triple $(A, \Delta_A, \mathcal{E}_A)$ is an *internal commutative comonoid*. Indeed, all relevant constraints like 'first copying and then deleting is the same thing as doing nothing', and similar ones, are all encoded in diagrams 5.1 and 5.2.

In a similar way, the relations

$$\delta = \{(x, (x, x)) \mid x \in X\} \subseteq X \times (X \times X) \quad \text{and} \quad \epsilon = \{(x, *) \mid x \in X\} \subseteq X \times \{*\}$$

defines an internal comoinoid on X in **Rel**. Indeed, using these ”copying” and ”deleting” operations makes diagrams 5.1 and 5.2 commute which the reader easily may verify.

For the category **FdHilb**, the comultiplication δ and the comultiplicative unit ϵ are used to encode bases. Indeed, given the basis

$$\mathcal{B} := \{|i\rangle \mid i = 1,...,n\}$$

of a Hilbert space $\mathcal{H}$, the two linear maps

$$\delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H} :: |i\rangle \longmapsto |ii\rangle \qquad \text{and} \qquad \epsilon : \mathcal{H} \longrightarrow \mathbb{C} :: |i\rangle \longmapsto 1$$

faithfully encode the basis $\mathcal{B}$ since we can extract it back from them. This amounts to solving the equation

$$\delta(|\psi\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for the unknown state $|\psi\rangle$. This equation is a product state if and only if the unknown state is the basis $\mathcal{B}$, for otherwise, for any other $\psi = \sum_i \alpha_i \, |i\rangle$, we have

$$\delta(|\psi\rangle) = \sum_i \alpha_i \, |i\rangle \otimes |i\rangle,$$

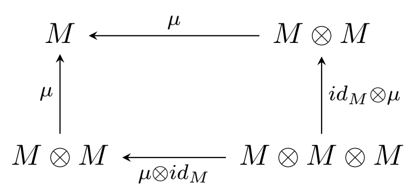
which is an *entangled* state.

The notion of internal comonoid can be dualized into the notion of *internal monoid*:

Definition (internal monoid). An *internal monoid* is an object $M \in |\mathbf{C}|$ together with a pair of morphisms

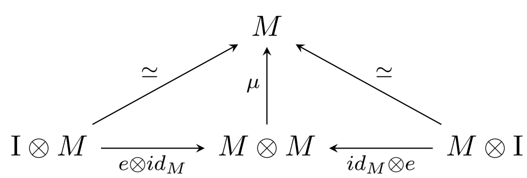
$$M \otimes M \xrightarrow{\mu} M \xleftarrow{e} \mathrm{I},$$

where μ is the *muliplication* and e is the *multiplicative* unit, which are defined through the commutation of the following two diagrams:



(5.4)

and,



(5.5)

The name ’internal monoid’ originates from the fact that ordinary monoids can equivalently be defined as being internal monoids in **Set**. Indeed, in **Set** the internal monoid (X, μ, e) is given by the functions

$$\mu : X \times X \longrightarrow X \qquad \text{and} \qquad e : \{*\} \longrightarrow X.$$

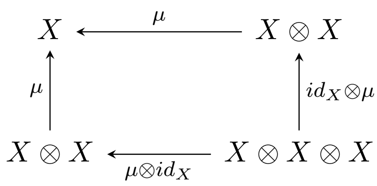
Now, in order to define the usual notion of a monoid we first need to make the following identification:

- the elements of the monoid are identified with the elements in X,
- the monoid operation is identified as

  $$- \bullet - : X \times X \longrightarrow X :: (x, y) \longmapsto \mu(x, y),$$

  and,
- the unit of the monoid is identified as $1 := e(*) \in X$.

Now, for **Set**, the diagram 5.4 becomes

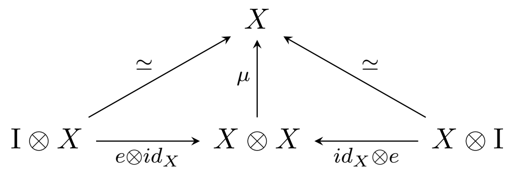


(5.6)

which, for the above identification, boils down to the fact that for all elements $x, y, z \in X$ we have

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z,$$

that is, it gives us the proper associativity requirement. On the other hand, diagram 5.2 becomes



(5.7)

which boils down to the fact that for all elements $x \in X$ we have

$$x \bullet 1 = 1 \bullet x = x,$$

that is, it gives us the proper unit requirement that all monoids share. Thus, the usual notion of a monoid can indeed be defined categorically as an internal monoid in **Set**.

In a similar fashion, the usual notion of a group can be stated categorically by introducing the notion of an *internal group*:

Definition (internal group). Let **C** be a category with finite products (see section 4.3.1) and let $\top$ be the terminal object in $|\mathbf{C}|$. An *internal group* is an internal monoid (G, μ, e) together with an additional morphism $\text{inv} : G \longrightarrow G$ which is such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\;!_G\;} & \top \\ {\scriptstyle \langle id_G, \text{inv}\rangle}\downarrow & & \downarrow{\scriptstyle e} \\ G \times G & \xrightarrow{\;\mu\;} & G \\ {\scriptstyle \langle \text{inv}, id_G\rangle}\uparrow & & \uparrow{\scriptstyle e} \\ G & \xrightarrow[\;!_G\;]{} & \top \end{array} \tag{5.8}$$

Where the additional morphism $inv : G \longrightarrow G$ assigns inverses to the elements of the group.

Again, using **Set** we can verify that the existence of an internal group in **Set** corresponds to the ordinary notion of a group. The internal monoid (X, μ, e) is again given by the functions

$$\mu : X \times X \longrightarrow X \qquad \text{and} \qquad e : \{*\} \longrightarrow X.$$

The additional morphisms which are used for making diagram 5.8 commute are given by the following functions:

$$\langle id_X, \text{inv}\rangle : X \longrightarrow X \times X \qquad \text{and} \qquad !_X : X \longrightarrow \{*\}.$$

Now, in order to define the usual notion of a group, we make an analogous identification as above, that is

- the elements of the group are identified with the elements in X,
- the group operation is identified as

  $$- \bullet - : X \times X \longrightarrow X :: (x, y) \longmapsto \mu(x, y),$$

  and,
- the unit of the group is identified as $1 := e(*) \in X$.

The proper associativity- and unit requirements are, again, satisfied through the commutation of diagrams 5.6 and 5.7 respectively. However, in order to verify that the inverses behave in accordance with the definition of a group, we need to verify that diagram 5.8 commutes. For **Set**, diagram 5.8 becomes:

$$\begin{array}{ccc} X & \xrightarrow{\;!_X\;} & \{*\} \\ {\scriptstyle \langle id_X, \text{inv}\rangle}\downarrow & & \downarrow{\scriptstyle e} \\ X \times X & \xrightarrow{\;\mu\;} & X \\ {\scriptstyle \langle \text{inv}, id_X\rangle}\uparrow & & \uparrow{\scriptstyle e} \\ X & \xrightarrow[\;!_X\;]{} & \{*\} \end{array} \tag{5.9}$$

Using that $* = \mu(x, x^{-1}) = \mu(x^{-1}, x)$ we have that the top square in 5.9 boils down to the fact that for all elements $x \in X$ we have

$$x \bullet x^{-1} = 1,$$

while the bottom square boils down to the fact that for all elements $x \in X$ we have

$$x^{-1} \bullet x = 1.$$

Thus, the usual notion of a group can indeed be defined categorically as an internal group in **Set**.

Since the concepts of internal monoids and internal groups applies to arbitrary monoidal categories, they generalise the usual notions of a monoid and a group respectively. In fact, the internal comonoids, internal monoids, and internal groups in a monoidal category can themselves be made into categories. The respective morphisms between them then generalises the usual notions of (co)monoid- and group homomorphisms.

The definition of a *group homomorphism* between two group objects (G, μ, e, inv) and $(G', \mu', e', \text{inv}')$ is a morphism $\phi : G \longrightarrow G'$ which is such that the following diagrams commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle \phi\times\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ G' \times G' & \xrightarrow[\mu']{} & G' \end{array} \quad , \quad \begin{array}{ccc} \top & \xrightarrow{e} & G \\ & \searrow^{e'} & \downarrow{\scriptstyle \phi} \\ & & G' \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\text{inv}} & G \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ G' & \xrightarrow[\text{inv}']{} & G' \end{array}$$

Again, these diagrams generalises the usual notion of group homomorphisms, that is, they preserve multiplication, unit and inverses respectively.

For monoids we instead have monoid homomorphisms which we define as follows:

Definition (Monoid homomorphism). A monoid homomorphism from a monoid (M, μ, e) to a monoid (M', μ', e') is a morphism $f : M \longrightarrow M'$ such that $f \circ \mu = \mu' \circ (f \otimes f)$ and $e' = f \circ e$.

Similarly, for comonoids we have comonoid homomorphisms which we define as follows:

Definition (Comonoid homomorphism). A comonoid homorphism from a comonoid (C, δ, ϵ) to a comonoid (C', δ', ϵ') is a morphism $g : C \longrightarrow C'$ such that $(g \otimes g) \circ \delta = \delta' \circ g$ and $(\epsilon' \circ g = \epsilon)$.

There are various ways in which a comonoid and a monoid on the same object can interact. In the following section we will develop one such way, namely *Frobenius algebras*.

5.2 Frobenius algebras

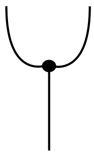
In a dagger monoidal category every internal comonoid

$$\left(A, A \xrightarrow{\delta} A \otimes A, A \xrightarrow{\epsilon} \mathrm{I}\right),$$

defines an internal monoid

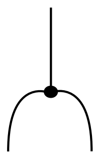
$$\left(A, A \xrightarrow{\delta^\dagger} A \otimes A, A \xrightarrow{\epsilon^\dagger} \mathrm{I}\right).$$

Indeed, this merely involves the reversal of arrows which we can depict graphically. We depict comultiplication and the comultiplicative unit as follows:

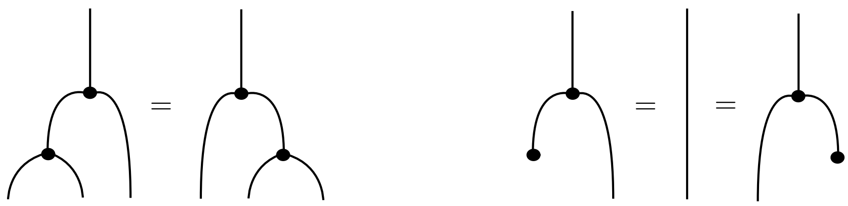
δ :=  ϵ := 

The corresponding conditions 5.4 and 5.5 are then depicted as:

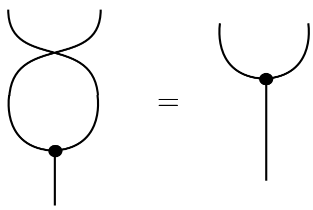
Now, if we turn these pictures upside-down we obtain a monoid:

$\delta^\dagger = \mu$:=  $\epsilon^\dagger = e$:= 

with the corresponding conditions 5.1 and 5.2 depicted as:



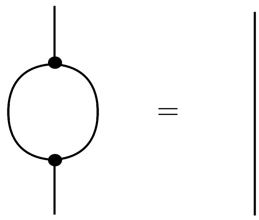
A *dagger (co)monoid* is a (co)monoid such that all the preceding requirements are satisfied. Thus, this is true for the dagger (co)monoids in **FdHilb** and **Rel** as well. However, as we have seen above, both **FdHilb** and **Rel** share some additional properties. For example, they are *commutative*:



that is, algebraically,

$$\sigma_{A,A} \circ \delta = \delta.$$

What is more, the comultiplication in both **FdHilb** and **Rel** is *special*, which is depicted as:

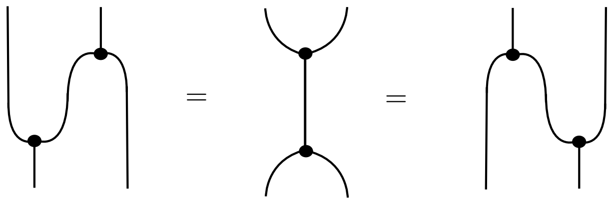


that is, algebraically,

$$\delta^\dagger \circ \delta = id_A.$$

Lastly, and most importantly, both **FdHilb** and **Rel** have (co)monoids in the form of so-called *dagger-Frobenius algebras*:

Definition (dagger-Frobenius algebra). In a dagger monoidal category, a *Frobenius algebra* is a comonoid (A, δ, ϵ) and a monoid $(A, \delta^\dagger, \epsilon^\dagger)$ satisfying the following law, called the *Frobenius equations*:



that is, algebraically,

$$(id_A \otimes \delta^\dagger) \circ (\delta \otimes id_A) = \delta \circ \delta^\dagger = (\delta^\dagger \otimes id_A) \circ (id_A \otimes \delta). \tag{5.10}$$

The Frobenius equations are named after F. Georg Frobenius in 1903 [55]. The formulation with multiplication and comultiplication we use is due to Lawvere in 1967 [56].

For a commutative dagger comonoid, it is easily seen that these two equations are equivalent. We will now verify that this is true for the particular dagger comonoids we introduced in **FdHilb** and **Rel** in the previous section.

In **FdHilb** we have:

$$\delta^\dagger : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} :: |ij\rangle \longmapsto \delta_{ij} \cdot |i\rangle \qquad \text{and} \qquad \epsilon^\dagger : \mathbb{C} \longrightarrow \mathcal{H} :: 1 \longmapsto \sum_i |i\rangle.$$

Thus, the first part of equation 5.10 for this particular comonoid becomes:

$$|ij\rangle \xmapsto{\delta \otimes id_{\mathcal{H}}} |iij\rangle \xmapsto{id_{\mathcal{H}} \otimes \delta^\dagger} |i\rangle \otimes (\delta_{ij} \cdot |i\rangle) = |ii\rangle$$

and

$$|ij\rangle \xmapsto{\delta^\dagger} \delta_{ij} \cdot |i\rangle \xmapsto{\delta} \delta_{ij} \cdot |ii\rangle.$$

In **Rel** we have

$$\delta^\dagger = \{((x,x),x) \mid x \in X\} \subseteq (X \times X) \times X \qquad \text{and} \qquad \epsilon^\dagger = \{(*,x) \mid x \in X\} \subseteq \{*\} \times X.$$

Thus, the first part of equation 5.10 for this particular comonoid becomes:

$$(id_X \otimes \delta^\dagger) \circ (\delta \otimes id_X) = \delta \circ \delta^\dagger = \{(x{,}x), (x{,}x)) \mid x \in X\}.$$

Commutative dagger-special Frobenius comonoids are not only abstract mathematical structures, they are also of great importance in quantum mechanics as exemplified by the following theorem:

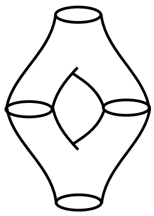
Theorem (Classical structures are bases). *In* **FdHilb**, *there is a bijective correspondence between dagger special Frobenius comonoids and orthonormal bases. Explicitly, each dagger special Frobenius comonoid in* **FdHilb** *is of the form*

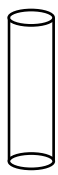
$$\delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H} :: |i\rangle \longmapsto |ii\rangle \qquad \text{and} \qquad \epsilon : \mathcal{H} \longrightarrow \mathbb{C} :: |i\rangle \longmapsto 1$$

relative to some orthonormal basis $\{|i\rangle\}_i$.

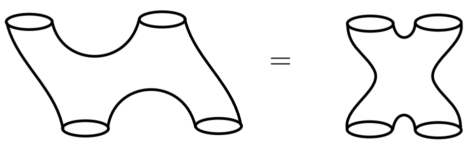
This theorem, that classical structures in **FdHilb** correspond to orthonormal bases, was proved in 2009 by Coecke, Pavlovic and Vicary in [57] where, in addition, more about dagger special comonoids and the relation between comonoids in **FdHilb** and quantum theory can be found.

In the category **2Cob** we also encounter (co)monoids. However, the (co)monoids in **2Cob** differs from those in **FdHilb** and **Rel** in that they do not constitute dagger special Frobenius algebras. The reason for this is that (co)monoids in **2Cob** are not special, since the two cobordisms





are not homeormorphic. However, (co)monoids in **2Cob** do constitute non-special dagger Frobenius algebras which implies that the following Frobenius equation hold:



There is a lot more to say about the connection between algebraic structures and the kind of pictures we have introduced in this section. For example, in the previous section we discussed how internal- groups and group homomorphisms in **Set** in fact are the usual notions of groups and group homomorphisms respectively. However, if we rather consider groups in other monoidal categories, in particular groups in monoidal categories of vector spaces, so-called *Hopf algebras* arises naturally, of which *quantum groups* are a special case. A quantum group is a vector space with structure. This structure is partly a multiplication and partly a comultiplication, which is dual to multiplication in the sense we described above. For more about quantum groups the reader may consult the following textbook by Ross Street, [58]. For more about Frobenius algebras in general we instead refer to [54] and [2].

We will come back to the concepts introduced in this section, particularly the notion of a Frobenius algebra, when we introduce the notion of topological quantum field theories in section 5.6. As will become apparent, the categorical restatement of a topological quantum field theory also depend crucially on notions regarding functors which will be the main topic of concern for the remaining part of this thesis.

5.3 Bifunctors

In section 2.2 we mentioned the concrete category **Cat** which has:

1. categories as objects,
2. functors between these as morphisms,
3. functor composition as morphism composition and,
4. identity functors as unit morphisms.

However, as we did for the Cartesian category, it is also possible to endow the concrete category **Cat** with monoidal structure:

Definition (**Cat** admits monoidal structure). The *product* of any two categories $\mathbf{C}$ and $\mathbf{D}$ is yet another category $\mathbf{C} \times \mathbf{D}$ for which:

1. objects are pairs (C, D) with $C \in |\mathbf{C}|$,
2. morphisms are pairs $(f, g) : (C, D) \longrightarrow (C', D')$ where $f : C \longrightarrow C'$ is a morphism in $\mathbf{C}$ and $g : D \longrightarrow D'$ is a morphism in $\mathbf{D}$,
3. composition of morphisms are component-wise, that is,
$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g),$$
4. and, unit morphisms are pairs of unit morphisms.

This monoidal structure is Cartesian. The projections functors are defined as:

$$\mathbf{C} \xleftarrow{P_1} \mathbf{C} \times \mathbf{D} \xrightarrow{P_2} \mathbf{D}$$

which provide the following product structure:

$$\begin{array}{ccccc} & & \forall \mathbf{E} & & \\ & \swarrow^{\forall Q} & \vdots\ \exists! F & \searrow^{\forall R} & \\ \mathbf{C} & \xleftarrow[P_1]{} & \mathbf{C} \times \mathbf{D} & \xrightarrow[P_2]{} & \mathbf{D} \end{array} \tag{5.11}$$

With product defined this way, *bifuncoriality* (see section 2.5) can be given a very concise definition. Indeed, a *bifunctor* is now nothing but an ordinary functor of type

$$F : \mathbf{C} \times \mathbf{D} \longrightarrow \mathbf{E}.$$

Indeed, we saw a concrete example of this fact in section 2.5 where showed that a monoidal product is nothing but a bifunctor. That is

$$- \otimes - : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

is a functor, which implies that we have

$$\otimes(f \circ g) = \otimes(f) \circ \otimes(g) \qquad \text{and} \qquad \otimes(1_C) = 1_{\otimes(C)}$$

for all morphisms f, g and all objects $C \in \mathbf{C} \times \mathbf{C}$.

In the following chapter we will develop the notion of natural transformations which essentially are maps going between functors.

5.4 Natural transformations

In section 3.4 we introduced the notion of a natural isomorphism which is a restricted versions of the more general notion of a *natural transformation* which we will develop in this section.

Recall that the natural isomorphisms we encountered in section 3.4 where given by:

$$\mathrm{I} \otimes A \simeq A \simeq A \times \mathrm{I} \quad , \quad A \otimes B \simeq B \otimes A \quad \text{and} \quad A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C.$$

The reason for why the above expressions are a restricted version of natural transformations is because they only involve objects in $\mathbf{C}$ without there being any reference to morphisms. For natural transformations, however, this is no longer the case. In fact, natural transformations are structure preserving maps between functors:

Definition (Natural transformation). Let $F, G : \mathbf{C} \longrightarrow \mathbf{D}$ be functors. A *natural transformation*

$$\tau : F \Rightarrow G$$

consists of a family of morphisms

$$\{\tau_A \in \mathbf{D}(FA, GA) \mid A \in |\mathbf{C}|\}$$

which are such that the following diagram

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\tau_B]{} & GB \end{array} \tag{5.12}$$

commutes for any objects $A, B \in |\mathbf{C}|$ and any morphism $f \in \mathbf{C}(A, B)$.

As an example of a natural transformation, let us consider $\mathbf{FdVect}_{\mathbb{K}}$, the category with finite dimensional vector spaces over the field $\mathbb{K}$ as objects and linear maps as morphisms. That is, let $F, G : \mathbf{FdVect}_{\mathbb{K}} \longrightarrow \mathbf{FdVect}_{\mathbb{K}}$ be the functors such that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{\tau_V} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\tau_V]{} & GB \end{array} \tag{5.13}$$

The natural transformation 5.13 then tell us that the vector spaces in $\mathbf{FdVect}_{\mathbb{K}}$ are basis independent. Indeed, the linear map $f : V \longrightarrow V$ can be interpreted as a change of basis, which then means that $Ff : FV \longrightarrow FV$ and $Gf : GV \longrightarrow GV$ apply this change of basis to the expressions FV and GV respectively. Commutation of diagram 5.13 then means that it makes whether we first apply τ_V before the change of basis, or whether we apply it after the change of basis. That is, the natural transformation 5.13 asserts that τ_V is a basis independent construction.

Having discussed functors and natural transformations, we are now in position to introduce monoidal functors and monoidal natural transformations which are of crucial importance for the categorical development of topological quantum field theories.

5.5 Monoidal functors and monoidal natural transformations

A monoidal functor is a map between monoidal categories that preserves the monoidal structure within them:

Definition (Monoidal functor). Let

$$(\mathbf{C}, \otimes, \mathrm{I}, \alpha_C, \lambda_C, \rho_C) \qquad \text{and} \qquad (\mathbf{D}, \odot, \mathrm{J}, \alpha_D, \lambda_D, \rho_D)$$

be monoidal categories. A *(lax) monoidal functor* then consist of the following information:

1. A functor
$$F : \mathbf{C} \longrightarrow \mathbf{D},$$

2. a morphism
$$\phi : \mathrm{J} \longrightarrow F\mathrm{I}, \quad \text{and,}$$

3. a natural transformation
$$\phi_{A,B} : (FA) \odot (FB) \Rightarrow F(A \odot B),$$
for all objects $A, B \in |\mathbf{C}|$.

A monoidal functor moreover satisfy the following conditions:

- (Associativity). For all objects $A, B, C \in |\mathbf{C}|$ the following diagram commutes

$$\begin{array}{ccc} (FA \odot FB) \odot FC & \xrightarrow{\alpha_D^{-1}} & FA \odot (FB \odot FC) \\ \downarrow{\scriptstyle \phi_{A,B} \odot id_{FC}} & & \downarrow{\scriptstyle id_{FA} \odot \phi_{B,C}} \\ F(A \otimes B) \odot FC & & FA \odot F(B \otimes C) \\ \downarrow{\scriptstyle \phi_{A \otimes B, C}} & & \downarrow{\scriptstyle \phi_{A, B \otimes C}} \\ F((A \otimes B) \otimes C) & \xrightarrow[F\alpha_C^{-1}]{} & F(A \otimes (B \otimes C)) \end{array} \tag{5.14}$$

- (Unitality). For all objects $A \in |\mathbf{C}|$ the following diagrams commute:

$$\begin{array}{ccc} FA \odot \mathrm{J} & \xrightarrow{id_{FA} \odot \phi} & FA \odot F\mathrm{I} \\ \downarrow{\scriptstyle \rho_D^{-1}} & & \downarrow{\scriptstyle \phi_{A,\mathrm{I}}} \\ FA & \xleftarrow[F\rho_C^{-1}]{} & F(A \otimes \mathrm{I}) \end{array} \quad , \quad \begin{array}{ccc} J \odot FB & \xrightarrow{\phi \odot id_{FB}} & F\mathrm{I} \odot FB \\ \downarrow{\scriptstyle \lambda_D^{-1}} & & \downarrow{\scriptstyle \phi_{\mathrm{I},B}} \\ FB & \xleftarrow[F\lambda_C^{-1}]{} & F(\mathrm{I} \otimes B) \end{array}$$

Moreover, a monoidal functor between symmetric monoidal categories is *symmetric* if, in addition, for all $A, B \in |\mathbf{C}|$ the following diagram commutes:

$$\begin{array}{ccc} FA \odot FB & \xrightarrow{\sigma_{FA,FB}} & FB \odot FA \\ \downarrow{\scriptstyle \phi_{A,B}} & & \downarrow{\scriptstyle \phi_{B,A}} \\ F(A \otimes B) & \xrightarrow[F\sigma_{A,B}]{} & F(B \otimes A) \end{array}$$

If all the natural transformations $\phi_{A,B}$ and the morphism ϕ are isomorphisms, then the monoidal functor F is called a *strong* monoidal functor. The monoidal functor F is called a *strict* monoidal functor if all the natural transformations $\phi_{A,B}$ and the morphism ϕ are identities. In this case the above conditions simplify to:

$$F(A \otimes B) = FA \odot FB \qquad \text{and} \qquad F\mathrm{I} = \mathrm{J}$$

and,

$$F\alpha_C = \alpha_D \;\; , \;\; F\lambda_C = \lambda_D \;\; , \;\; F\rho_C = \rho_D \;\; \text{and} \;\; F\sigma_C = \sigma_D .$$

Hence, a strict monoidal functor between strict monoidal categories just means that the tensor is preserved.

A monoidal natural transformation is a natural transformation between monoidal functors that respects the monoidal structure:

Definition (Monoidal natural transformation). A *monoidal natural transformation*

$$\Gamma : (F, \{\phi_{A,B} \mid A, B \in |\mathbf{C}|, \phi) \Rightarrow (G, \{\psi_{A,B} \mid A, B \in |\mathbf{C}|, \psi)$$

between two monoidal functors is a natural transformation such that the following diagrams commute:

$$\begin{array}{ccc} FA \odot FB & \xrightarrow{\Gamma_A \odot \Gamma_B} & GA \odot GB \\ \downarrow{\scriptstyle \phi_{A,B}} & & \downarrow{\scriptstyle \psi_{A,B}} \\ F(A \otimes B) & \xrightarrow[\Gamma_{A \otimes B}]{} & G(A \otimes B) \end{array} \quad \text{and} \quad \begin{array}{ccc} & \mathrm{J} & \\ {\scriptstyle \phi} \swarrow & & \searrow {\scriptstyle \psi} \\ F\mathrm{I} & \xrightarrow[\Gamma_{\mathrm{I}}]{} & G\mathrm{I} \end{array}$$

Moreover, the monoidal natural transformation is *symmetric* if it maps between two symmetric monoidal categories.

We have now developed category theory enough so as to be able to give a categorical definition of a generic n-dimensional topological quantum field theory which will be the final result we present in this project.

5.6 Topological quantum field theories

In this section we will finally present a categorical construction of topological quantum field theories (TQFTs). In fact, the categorical definition of a generic n-dimensional TQFT is nothing but a monoidal functor between the categories **nCob** and $\mathbf{FdVect}_{\mathbb{K}}$. The definition is very elegant and simple. However, the seemingly short definition that we present below is packed with subtleties. For this reason, we will first give the non-categorical axioms of generic n-dimensional TQFTs as stated by Atiyahs's definition, which can be found in [53]. We then show how these axioms can be reformulated in categorical terms. A more elaborate discussion regarding the categorical restatement of TQFTs can be found in [54] while a discussion regarding the connection between the categorical restatement of TQFTs and quantum gravity can be found in [52], [11] and [43].

Definition (Atiyah's definition of TQFTs). An n-*dimensional TQFT* is a rule Z which assign a vector space $Z(S)$ to each $(n-1)$-dimensional manifold S over the field $\mathbb{K}$, and to each oriented map cobordism $M : S_0 \longrightarrow S_1$, a linear map $Z(M) : M(S_0) \longrightarrow M(S_1)$ subjected to the following conditions:

1. if two cobordisms are homoemorphic $M \simeq M'$ then we have $Z(M) = Z(M')$,
2. identities are preserved, that is, for any manifold S, we have
$$Z(id_S) = id_{Z(S)},$$
where the right-hand side denotes the identity map on the vector space $Z(S)$,
3. if $M = M' \circ M''$ then we have
$$Z(M) = Z(M') \circ Z(M''),$$
4. the disjoint union $S = S' + S''$ is mapped to
$$Z(S) = Z(S') \otimes Z(S''),$$
and the disjoint union $M = M' + M''$ is mapped to
$$Z(M) = Z(M') \otimes Z(M''), \text{ and,}$$

5. the empty manifold $S = \emptyset$ is mapped to the ground field $\mathbb{K}$ and the empty cobordism is sent to the identity map on $\mathbb{K}$.

The above five axioms can now be compressed into a very short and elegant definition:

Definition (TQFTs as a monoidal functor). An n-*dimensional TQFT* is a symmetric monoidal functor

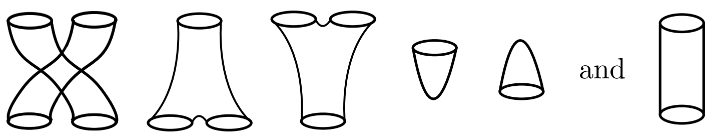
$$Z : (\mathbf{nCob}, +, \emptyset, T) \longrightarrow (\mathbf{FdVect}_{\mathbb{K}}, \otimes, \mathbb{K}, \sigma)$$

, where T are the twist cobordisms we developed in section 4.2.2.

The rule Z which is used to map cobordisms M to linear maps $Z(M)$ and which maps manifolds S to vector spaces $Z(S)$ constitute the domain and codomain for the symmetric monoidal functor. Axiom 1 above tells us that we consider homeomorphism classes of cobordisms, that is, cobordisms which are deformable. Axioms 2 and 3 constitute the definition of a functor. Axioms 4 and 5 moreover tells us that we are dealing with a monoidal functor.

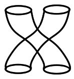
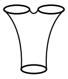
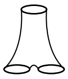
We next show an example of how such a monoidal functor explicitly can be constructed. In order to do so we will consider the category **2Cob** together with the following proposition which can be found in [54]:

Proposition. *The monoidal category* **2Cob** *is generated by:*



That is, any cobordism in **2Cob** *can be expressed in terms of these generators using composition and the monoidal product.*

Note that the above generators satisfy the axioms of a Frobenius comonoid (see section 5.2). Thus, in order to specify the monoidal functor Z, it is sufficient to give the image of the generators in **2Cob**. That is, the monoidal functor Z is a map from this Frobenius comonoid in **2Cob** on a Frobenius comonoid in $\mathbf{FdVect}_{\mathbb{K}}$:

Objects:	n	$\longmapsto$	$\underbrace{V \otimes V \otimes \ldots \otimes V}_{n \text{ times}}$
Identity:		$\longmapsto$	$id_V : V \longrightarrow V$
Twist:		$\longmapsto$	$\sigma_{V,V} : V \otimes V \longrightarrow V \otimes V$
e :		$\longmapsto$	$e : \mathbb{K} \longrightarrow V$
ϵ :		$\longmapsto$	$\mu : V \otimes V \longrightarrow V$
δ :		$\longmapsto$	$\epsilon : V \longrightarrow \mathbb{K}$
μ :		$\longmapsto$	$\delta : V \longrightarrow V \otimes V$

The converse is true as well, that is, it is also possible to define a TQFT by the preceding prescription given a Frobenius comonoid in V, so there is a one-to-one correspondence between commutative Frobenius comonoids and 2-dimensional TQFTs. In fact, it turns out that Frobenius comonoids and 2-dimensional TQFTs are even more deeply linked to each other.

Indeed, we can now define $\mathbf{2TQFT}_{\mathbb{K}}$, the category with 2-dimensional TQFTs as objects and symmetric monoidal natural transformations between them as morphisms. From the definition of natural transformations we presented in section 5.5, we have that: given two TQFTs $Z, Z' \in |\mathbf{2TQFT}_{\mathbb{K}}|$, the natural transformation Γ is given by

$$\Gamma_n : \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} \longrightarrow \underbrace{W \otimes W \otimes \cdots \otimes W}_{n \text{ times}}.$$

Since this natural transformation is monoidal, it is completely specified by the map $\Gamma_1 : V \longrightarrow W$. The morphism $\Gamma_{\mathbb{K}}$ is the identity mapping from the trivial Frobenius comonoid on $\mathbb{K}$ to itself. Finally, naturality of Γ means that the components must commute with the morphisms of **2Cob**. However, since **2Cob** can be decomposed into the generators above we only have to consider these cobordisms, for example:

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\Gamma_2} & W \otimes W \\ \downarrow{\scriptstyle \mu_V} & & \downarrow{\scriptstyle \mu_W} \\ V & \xrightarrow[\Gamma_1]{} & W \end{array}$$

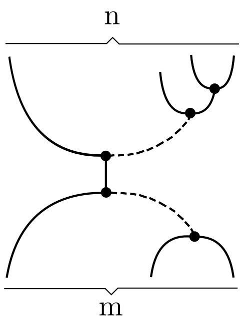
We can now define $\mathbf{CFC}_{\mathbb{K}}$, the category of commutative Frobenius comonoids and morphisms of Frobenius comonoids, that is, linear maps that are both comonoid homomorphisms and monoid homomorphisms (see section 5.2). The reason why we introduce $\mathbf{CFC}_{\mathbb{K}}$ is due to the following theorem which was discovered by Dijkgraaf, who realized in 1989 that the category of commutative Frobenius algebras is equivalent to that of 2-dimensional topological quantum field theories [59]:

Theorem. *The category* $\mathbf{2TQFT}_{\mathbb{K}}$ *is equivalent to the category* $\mathbf{CFC}_{\mathbb{K}}$.

For a comprehensive treatment regarding this theorem, see the monograph by Kock [60].

The above theorem is a prime example of how category theory can be used to study physical theories in which TQFTs are of importance. Indeed, the mathematics of Frobenius comonoids is in general much simpler than the corresponding mathematics of TQFTs. A case in point is the so-called *spider theorem*:

Theorem (Spider theorem). *Let* (A, δ, ϵ) *be a classical structure. Any connected morphism* $A^{\otimes m} \longrightarrow A^{\otimes n}$ *built out of* $\delta, \epsilon, id, \sigma, \otimes$ *and* $\dagger$ *equals the following normal form.*



So any morphism built from $\delta, \epsilon, id, \sigma, \otimes$ *can be built from normal forms with* $\otimes$ *and* σ*.*

One can show that the spider theorem follows from the Frobenius equation together with speciality, [61], [62]. Thus, commutative dagger special Frobenius comonoids turn out to be structures which only depends on the number of inputs and output wires and which moreover come with a very simple graphical calculus.

6

Conclusions and Discussion

This concludes our introductory tutorial of (a small part of) category theory. We have primarily focused on monoidal categories which provides a powerful starting point for building physical models. This is because monoidal categories naturally embody the structure of physical systems and processes which was demonstrated by the development of distinct monoidal structures which we used to indicate some key differences between classical- and quantum physical systems. However, this is not to say that the distinction as presented in this thesis in any way gives the full picture of what quantum theory is truly about. Indeed, all of this is part of a novel and vastly growing research area for which we hope this project may serve as a suitable introduction. More elaborate discussions regarding categorical comparisons between classical and quantum theories can be found in [51] and [63].

One of the main results of this thesis, which is a direct consequence of the difference in categorical structure within the Cartesian category (classical) and the dagger-compact category (quantum), was the categorical proof of the no-cloning theorem of quantum information theory. That is, we showed that objects in the dagger-compact category **FdHilb** cannot be copied. What is particularly interesting is that the categorical proof not only explains that we cannot copy quantum states but also *why* we cannot do so from a categorical perspective. We thus argued that it may be more enlightening, from the perspective of *logic*, to study quantum theory from the perspective of the dagger-compact category **FdHilb** (or some other 'quantum category') rather than the more conventional mathematical approach, namely, set theory.

This idea of using monoidal categories to better understand the foundation of quantum mechanics stems from the widespread sense that the principles behind quantum theory are poorly understood compared to those of general relativity which has led to many discussions about interpretational issues. The urge for a more logical quantum theory led physicists to develop the research field known as *quantum logic* which attempts to cast more light on the puzzle of quantum mechanics from the point of view of logic. Indeed, ever since the very influential 1936 paper by Birkhoff and von Neumann entitled, "The logic of quantum mechanics" [64], quantum logic has made considerable progress towards understanding the field of candidates from which the particular formalism we use has been chosen. However, quantum logic does not yet explained why the particular formalism we use is so special. For example, why do we use complex Hilbert spaces rather than real or quaternionic ones? Is there a deep reason for this or is it only because this decision fit experimental data? The study of particular monoidal categories, or more precisely, the 'internal structure' of particular monoidal categories, can be thought of as a modern approach to the structure of quantum logic where these type of questions are at the forefront.

We have briefly discussed the topic of 'internal classical structures' in this thesis. In category theory, any category has its own 'internal logic' which gives us a opportunity to study their logic from a different perspective than is attained by studying them externally - which gives us a classical logic perspective. That is, it is also possible to study non-classical internal structures of which the categories **FdHilb** and **nCob** are two examples. However, the textbook treatments and even most of the research literature on categorical logic focus on categories where the monoidal structure is Cartesian (classical). The study of logic within more general monoidal categories is at its infancy, however, there are at least few papers which do present work regarding more general internal structures, see for example Abramsky and Coecke (2004) [29].

An interesting idea regarding all of this is that the structural similarities between the categories **FdHilb**, which has to do with quantum theory, and **nCob**, which has to do with general relativity, hints toward a possible unification of these two realms. For a more elaborate but still pedagogical discussion regarding this idea, see John Baez (2004) [43]. The main observation in his paper is that **FdHilb**, structurally, is much more reminiscent of **nCob** than it is to **Set**. Now, we usually make use of the mathematics of **Set** in quantum theory. However, the structural similarities between **FdHilb** and **nCob** implies that, perhaps, quantum theory will make more sense when (or if) quantum theory is reconciled with general relativity.

Of course, there is still a lot more work do be done before we have a theory of quantum gravity. As of today, one promising approach towards such a theory is to study topological quantum field theories (TQFTs) which are *background-free quantum theories with no local degrees of freedom*. Although TQFTs in this regard are not sophisticated enough to explain the universe we live in, they do present a working model from which calculations gives insights into a more full-fledged theory of quantum gravity. That is, they possess certain features one would expect a theory of quantum gravity to possess. For a non-categorical discussion regarding TQFTs and their role within fundamental physics, see Robbert Dijkgraaf's paper entitled 'Les Houches Lectures on Fields, Strings and Duality*' (1995) [12] and John W. Barrett's paper entitled 'Quantum Gravity as Topological Quantum Field Theory' (1995) [13].

From a category-theoretic perspective, what is interesting is that the study of TQFTs increasingly make use of so-called 'n-categories'. While categories consists of objects and morphisms between these objects, n-categories also have 2-morphisms between morphisms and 3-morphisms between 2-morphisms and so on to the n:th degree. One way to visualize this is to imagine the objects of the category as points, the morphisms as arrows going between these points, the 2-morphisms as 2-dimensional surfaces going between these arrows and so on. There is thus a natural link between n-categories and n-dimensional topology, the latter of which category theorists often use the category **nCob** to describe. However, this link between n-categories and n-dimensional topology suggests that we instead can use the simpler mathematics of n-categories to describe **nCob** which is precisely why category theorists use n-categories to study higher dimensional TQFTs.

In this project we have shown how the mathematical definition of a TQFT can be cast into categorical language in a manner which explicitly takes the categories $\mathbf{FdVect}_{\mathbb{K}}$ and

nCob into account. However, we have not discussed the utility of using n-categories to study them for which we refer to the following paper by John Baez and James Dolan entitled 'Higher-Dimensional Algebra and Topological Quantum Field Theory' [52].

We end by recommending some other great references. The first source we would like to recommend is a book by Jiri Adámek, Horst Herrlich and George E. Strecker entitled 'The Joy of Cats', [18]. This book explains category theory from the bottom up in a pedagogical manner and has been very inspirational to this thesis. The book also contains a chapter regarding *adjoint functors* which are, from the point of view of the mathematician, the greatest achievement of category theory thus far. Adjoint functors are used to essentially unify all known mathematical constructs of a variety of areas of mathematics including but not limited to algebra, geometry, topology, analysis and combinatorics, all within a single mathematical concept.

Next we would like to recommend an article by Peter Selinger entitled 'A survey of graphical languages for monoidal categories', [2]. This article presents a more general and sophisticated treatment regarding the graphical notation of monoidal categories we introduced in this project.

Lastly we would also like to recommend a compilation of notes presented by Samson Abramsky and Nikos Tzevelekos in the paper entitled 'Introduction to Categories and Categorical Logic', [44]. These notes provides essential ideas regarding category theory as to introduce the reader towards the fascinating research area known as *categorical logic* which we briefly discussed above. In addition to being a great introduction towards categorical logic, these notes also contain further important categorical concepts which we did not mention in this project but are of great relevance for constructing more sophisticated physical models from category theory, for example, a categorical model regarding anyons in relation to quantum computing (see [6]).

Bibliography

[1] S. Eilenberg, S. MacLane, *General theory of natural equivalence*, Transactions of the American Mathematical Society **58** (1945) 231–294.

[2] P. Selinger, *A survey of graphical languages for monoidal categories*, Springer Lecture Notes in Physics **813** (2011) 289–355.

[3] P. Panangaden, Éric Oliver Paquette, *A categorical presentation of quantum computation with anyons*, Available at www.cs.mcgill.ca/ prakash/Pubs/MTCanyons (2011).

[4] C. Nayak, S. H. Simon, A. Stern, M. Freedman, S. D. Sarma, *Non-Abelian Anyons and Topological Quantum Computation*, Reviews of Modern Physics **80** 1083 (2008).

[5] F. Wilczek, *Magnetic Flux, Angular Momentum, and Statistics*, Physical Review Letters **48** 1144 (1982).

[6] A. Blass, Y. Gurevich, *On quantum computation, anyons, and categories*, arXiv:1502.00669v3 [quant-ph] (2015).

[7] J. C. Baez, J. D. Biamonte, *Quantum Techniques for Stochastic Mechanics*, arXiv:1209.3632v2 [quant-ph] (2015).

[8] P. Wadler, *Comprehending Monads*, Mathematical Structures in Computer Science 2 (1992) 461–493.

[9] S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, Handbook of Logic in Computer Science: Logic and algebraic methods, Oxford University Press, 2011.

[10] M. Barr, C. Wells, Category theory for computing science, available at www.math.mcgill.ca/triples/barr-wells-ctcs (1990).

[11] J. C. Baez, *Higher-Dimensional Algebra and Planck-Scale Physics*, arXiv:gr-qc/9902017v1 (1999).

[12] R. Dijkgraaf, *Les Houches Lectures on Fields, Strings and Duality*, arXiv:hep-th/9703136v1 (1997).

[13] J. W. Barrett, *Quantum Gravity as Topological Quantum Field Theory*, journal of mathematical physics **36** (1995) 6161-6179.

[14] J. Kock, *Frobenius algebras and 2D topological quantum field theories*, Available at http://mat.uab.es/ kock/TQFT.html (2003).

[15] V. G. Turaev, *Axioms for topological quantum field theories*, Annales de la faculte des sciences de Toulouse 6 (**3**) (1994) 135–152.

[16] B. Coecke, Éric Oliver Paquette, *Categories for the practising physicist*, arXiv:0905.3010v2 [quant-ph] (2009).

[17] B. Coecke, *Introducing categories to the practicing physicist*, arXiv:0808.1032v1 [quant-ph] (2008).

[18] H. H. Jiri Adámek, G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Available at: http://katmat.math.uni-bremen.de/acc/ (2004).

[19] S. Abramsky, B. Coecke, *Categorical Quantum Mechanics*, Available at http://www.indiana.edu/ iulg/qliqc/ (2009).

[20] S. MacLane, Categories for the Working Mathematician, Springer Verlag, 2000.

[21] R. Goldblatt, *Topoi, the categorial analysis of logic (2002)*, Available at Cornell University Library: http://ebooks.library.cornell.edu/cgi/t/text/text-idx?c=math;cc=math;view=toc;subview=short;idno=Gold010.

[22] D. I. Spivak, *Category theory for scientists (Old version)*, arXiv:1302.6946v3 [math.CT] (2013).

[23] F. W. Lawvere, S. H. Schanuel, Conceptual Mathematics A First Introduction to Categories 2nd Edition, Cambridge University Press; 2 edition (2009).

[24] S. M. Lane, *Duality for groups*, Bulletin of the American Mathematical Society, 56(6):485–516, (1950).

[25] P. H. Palmquist, *Adjoint functors induced by adjoint linear transformations*, Proceedings of the American Mathematical Society, 44(2):251–254, (1974).

[26] P. Selinger, *Idempotents in dagger categories*, In Quantum Programming Languages, volume 210 of Electronic Notes in Theoretical Computer Science, pages 107– 122. Elsevier, (2008).

[27] R. Penrose, *Applications of negative dimensional tensors*, Combinatorial Mathematics and its Applications, 221-244, Academic Press, (1971).

[28] A. Joyal, R. Street, *The geometry of tensor calculus I*, Advances in Mathe- matics, 88:55–113, (1991).

[29] S. Abramsky, B. Coecke, *A Categorical semantics of quantum protocols*, Proceedings of the 19th IEEE conference on Logic in Computer Science (LiCS'04). IEEE Computer Science Press (2004).

[30] C. Kuratowski, *Sur la notion de l'ordre dans la Théorie des Ensembles*, Fundamenta Mathematicae. 2 (1921) (1): 161–171.

[31] M. Acclavio, *A complete proof of coherence for symmetric monoidal categories using rewriting*, arXiv:1606.01722v1 [math.CT] 2016.

[32] J. Bénabou, *Categories avec multiplicatio*, Categories avec multiplication. Comptes Rendus de l'Acadmie des Sciences. Série I. Mathmatique, pages 1887–1890, (1963).

[33] S. M. Lane, *Natural associativity and commutativity*, Rice University Studies, 49:28–46, (1963).

[34] S. Abramsky, *No-Cloning In Categorical Quantum Mechanics*, Available at http://www.indiana.edu/ iulg/qliqc/ (2009).

[35] G. Kelly, M. Laplaza, *Coherence for compact closed categories*, Journal of Pure and Applied Algebra (19) (1980) 193–213.

[36] B. C. Samson Abramsky, *A categorical semantics of quantum protocols*, arXiv:quant-ph/0402130v5 (2004).

[37] B. Coecke, *Kindergarten Quantum Mechanics*, arXiv:quant-ph/0510032v1 (2005).

[38] G. Boole, *An Investigation of the Laws of Thought (1854)*, Prometheus Books, (2007).

[39] D. R. Finkelstein, D. S. J. M. Jauch, D. Speiser, *Foundations of quaternion quantum mechanics*, Journal of Mathematical Physics (3) (1962) 207–220.

[40] S. L. Adler, *Quaternion Quantum Mechanics and Quantum Fields*, Oxford U. P. (1995).

[41] A. Samson, G. Dov, *Handbook of Logic in Computer Science: Logic and algebraic methods*, Oxford: Oxford University Press. ISBN 0-19-853781-6 (2001).

[42] G. M. Kelly, *Many variable functorial calculus (I)*, In Coherence in Categories, volume 281 of Lecture Notes in Mathematics, pages 66–105. Springer, (1970).

[43] J. C. Baez, *Quantum quandaries: A category-theoretic perspective*, in S. French et al. (Eds.) Structural Foundations of Quantum Gravity, Oxford University Press, 2004. arXiv:quant-ph/0404040.

[44] S. Abramsky, N. Tzevelekos, *Introduction to categories and categorical logic*, New Structures for Physics, B. Coecke (ed), Springer Lecture Notes in Physics, 2009.

[45] J. S. Bell, *On the Einstein-Podolsky-Rosen paradox*, Physics 1 (1964), 195–200.

[46] J. Baez, *Bell's inequality for C*-algebras*, Math. Phys. 13 (1987), 135–136.

[47] W. K. Wootters, W. H. Zurek, *A single quantum cannot be cloned*, Nature 299 (1982), 802–803.

[48] W. K. Wootters, W. H. Zurek, *A single quantum cannot be cloned*, Nature, 299:802–803, (1982).

[49] D. Dieks, *Communication by EPR devices*, Physics Letters A, 92(6):271–272, (1982).

[50] S. Abramsky, *No-cloning in categorical quantum mechanics*, In Simon Gay and Ian Mackey, editors, Semantic Techniques in Quantum Computation, pages 1–28. Cambridge University Press, (2010).

[51] B. B.Coecke, R.W.Spekkens, *The group-theoretic origin of non-locality for qubits*, Oxford University Computing Laboratory Research Report PRG-RR-09-04, (2009).

[52] J. D. John C. Baez, *Higher-dimensional Algebra and Topological Quantum Field Theory*, arXiv:q-alg/9503002v2 (2004).

[53] M. Atiyah, *The Geometry and Physics of Knots, Cambridge U. Press, Cambridge*, Topological quantum field theories Inst. Hautes E tudes Sci. Publ. Math. 68 (1988) 175-186.

[54] J. Kock, *Frobenius Algebrasand 2D Topological Quantum Field Theories*, London Mathematical Society, in Student Texts 59, (2004).

[55] F. Frobenius, *Theorie der hyperkomplexen grössen*, Sitzungsberichteder Koniglich Preussischen Akademie Der Wissenschaften, 24:504–537; 634–645, (1903).

[56] F. W. Lawvere, *Ordinal sums and equational doctrines*, In Beno Eckmann, editor, Seminar on triples and categorical homology theory, number 80 in Lecture Notes in Mathematics, pages 141–155, (1967).

[57] J. V. B. Coecke, D. Pavlovic, *A new description of orthogonal bases*, arXiv:0810.0812 [quant-ph] (2008).

[58] R. Street, *Quantum Groups: A Path to Current Algebra*, Cambridge University Press, (2007).

[59] R. Dijkgraaf, *A geometric approach to two dimensional conformal field theory*, PhD thesis, University of Utrecht, (1989).

[60] G. M. Kelly, M. L. Laplaza, *Coherence for compact closed categories*, Journal of Pure and Applied Algebra, 19:193–213, (1980).

[61] S. Lack, *Composing PROPs*, Theory and Applications of Categories 13, 147–163, (2004).

[62] B. Coecke, E. O. Paquett, *POVMs and Nairmarks theorem without sums*, Electronic Notes in Theoretical Computer Science (to appear), 2008. arXiv:quant-ph/ 0608072.

[63] B. Coecke, *New structures for physics*, Number 813 in Lecture Notes in Physics. Springer, (2011).

[64] G. Birkhoff, J. vonn Neumann, *"The Logic of Quantum Mechanics"*, Available at: Ann. Math. 37 (4): 823–843. JSTOR 1968621.

A

In this Appendix we show, using Kuratowskis's definition of ordered pairs, that

$$(x,(y,z)) \neq ((x,y)\,,z) \qquad \text{and} \qquad (x,*) \neq x \neq (*,x)\,.$$

Kuratowski's definition is given by

$$(a,b) = \Big\{\{a\}\,,\{a,b\}\Big\}\,,$$

where (a,b) is a set of ordered pairs while $\{a,b\}$ is a set of unordered pairs. This definition serves the purpose of assuring that the defining property of ordered pairs

$$(a,b) = (c,d) \text{ if and only if } a = c \text{ and } b = d,$$

is satisfied.

Now, using Kuratowski's definition we have

$$\Big((x,y),z\Big) = \Big(\{x\}\,,\{x,y\}\,,z\Big) = \Big\{\{\{x\}\,,\{x,y\}\},\{\{x\}\,,\{x,y\}\,,z\}\Big\}$$

while

$$\Big(x,(y,z)\Big) = \Big(x,\{\{y\},\{y,z\}\}\Big) = \Big\{\{x\},\{x,\{\{y\},\{y,z\}\}\}\Big\}\,.$$

Since the elements are ordered differently, these are indeed different sets. In an analogues way we also have that $(x,*) \neq x \neq (*,x)$.

B

In this appendix we explicitly verify that the the category **Rel** is †-compact. We will perform this calculation step by step. That is, we will first make sure that **Rel** can be made into a symmetric monoidal category before we show that it can be made into a compact category and lastly, we verify that it is †-compact as well. We will only consider the relevant coherence conditions. The results regarding all relevant naturality conditions are obtained in a similar fashion.

Recall from section 4.2.1 that the symmetric monoidal category contains the following information:

1. Sets are identified as objects.
2. Any singleton $\{*\}$ is identified with the monoidal unit I.
3. Relations of type $R : X \longrightarrow Y$ are identified as morphisms.
4. For $R_1 : X \longrightarrow Y$ and $R_2 : Y \longrightarrow Z$ the composite $R_2 \circ R_1 \subseteq X \times Z$ is given by
$$R_2 \circ R_1 := \{(x,z) \mid \text{there exists a } y \in Y \text{such that } xR_1y \text{ and } yR_2z\}.$$
5. The Cartesian product between sets is identified with the monoidal product between objects such that, for $R_1 : X_1 \longrightarrow Y_1$ and $R_2 : X_2 \longrightarrow Y_2$ the monoidal product is given by
$$R_1 \times R_2 := \{((x,x'),(y,y')) \mid xR_1y \text{ and } x'R_2y'\} \subseteq (X_1 \times X_2) \times (Y_1 \times Y_2). \tag{B.1}$$
6. The left- and right unit natural isomorphisms are respectively given by
$$\lambda_X := \{(x,(*,x)) \mid x \in X\} \qquad \text{and} \qquad \rho_X := \{(x,(x,*)) \mid x \in X\}.$$
7. The associativity natural isomorphism is given by
$$\alpha_{X,Y,Z} := \{((x,(y,z)),((x,y),z) \mid x \in X,\ y \in Y \text{ and } z \in Z\}.$$

8. For any X and $Y \in |\mathbf{Rel}|$, the symmetry natural isomorphism is given by

$$\sigma_{X,Y} := \{((x,y),(y,x)) \mid x \in X \text{ and } y \in Y\}.$$

In order to verify that this information suffice to make **Rel** into a symmetric monoidal category, the coherence conditions 3.14, 3.15, 3.16, 3.17 and 3.18 (see section 3.4) must all commute.

What is more, the natural isomorphisms $\lambda_X, \rho_X, \alpha_{X,Y,Z}$ and $\sigma_{X,Y}$ in **Rel** above are all single-valued relations, so they are also functions. Indeed, a function is a relation between two sets with the condition that for each element in the domain there is only image in the codomain. So, since the relations $\lambda_X, \rho_X, \alpha_{X,Y,Z}$ and $\sigma_{X,Y}$ in **Rel** relates exactly on element $x \in X$ to one element $y \in Y$ they are also functions. These functions are in fact the same as the natural isomorphisms for the Cartesian product in **Set**, so by verifying that **Rel** constitutes a symmetric monoidal category we simultaneously also verify that **Set** is such as well.

Let us begin by verifying that the Mac Lane pentagon 3.14

$$\begin{array}{ccccc} W \times (X \times (Y \times Z)) & \xrightarrow{\alpha_{\_}} & (W \times X) \times (Y \times Z) & \xrightarrow{\alpha_{\_}} & ((W \times X) \times Y) \times Z \\ \big\downarrow{\scriptstyle id \times \alpha_{\_}} & & & & \big\downarrow{\scriptstyle \alpha_{\_} \times id} \\ W \times ((X \times Y) \times Z) & & \xrightarrow[\alpha_{\_}]{\qquad\qquad\qquad} & & (W \times (X \times Y)) \times Z \end{array} \tag{B.2}$$

commutes. The top part of the coherence diagram is given by

$$\alpha_{\_} \circ \alpha_{\_} : W \times (X \times (Y \times Z)) \longrightarrow ((W \times X) \times Y) \times Z.$$

Using clause 4 above, this composition is by definition a subset of

$$(W \times (X \times (Y \times Z)) \times ((W \times X) \times Y) \times Z),$$

which is given by

$$\begin{aligned} \alpha_{\_} \circ\ \alpha_{\_} = \Big\{ &((w,(x,(y,z))),(((w'',x''),y'',z'')) \ \Big|\ \exists\, ((w',x'),(y',z') \text{ s.t} \\ &(w,(x,(x,y)))\alpha_{\_}((w',x'),(y',z')) \text{ and } ((w',x'),(y',z'))\alpha_{\_}(((w'',x''),y''),z'') \Big\}, \end{aligned} \tag{B.3}$$

which by the definition of α simplifies to

$$\alpha_{\_} \circ\ \alpha_{\_} = \{((w,(x,(y,z))),(((w,x,y),z)) \mid w \in W, x \in X, y \in Y, z \in Z\}.$$

The same result is obtained, in an analogous way, when considering the bottom path, making the Mac Lane pentagon commute.

Let us now verify that the triangle coherence condition given by 3.15 commutes as well:

$$\begin{array}{ccc} X \times Y & \xrightarrow{id_X \times \lambda_Y} & X \otimes (\{*\} \times Y) \\ & \searrow^{\rho_X \times id_Y} & \downarrow \alpha_{X,\{*\},Y} \\ & & (X \times \{*\}) \times Y \end{array} \tag{B.4}$$

The top part of the coherence diagram is given by

$$\alpha_{X,\{*\},Y} \circ id_X \times \lambda_Y : X \times Y \longrightarrow (X \times \{*\}) \times Y.$$

By making use of claus 4, this composition is by definition a subset of

$$(X \times Y) \times ((X \times \{*\}) \times Y)$$

which is given by

$$\alpha_{X,\{*\},Y} \circ id_X \times \lambda_Y = \Big\{(x,y),((x'',\{*\}),y'') \;\Big|\; \exists (x',(\{*\},y')) \text{ s.t} \\ (x,y) id_X \times \lambda_Y (x',(x',(\{*\},y') \text{ and } (x',(\{*\},y') \alpha_{X,\{*\},Y}((x'',\{*\}),y'')\Big\}, \tag{B.5}$$

which by the definition of id, λ and α simplifies to

$$\alpha_{X,\{*\},Y} \circ id_X \times \lambda_Y = \{(x,y),((x,\{*\},y) \mid x \in X, * \in \{*\}, y \in Y\}.$$

The bottom path yields the same result, hence making the triangle commute.

With these verification's we have that both **Rel** and **Set** constitutes monoidal categories. In order to verify that they also are symmetric, we need that the coherence conditions 3.16, 3.17 and 3.18 commutes as well. Let us start by considering the triangles 3.16 and 3.17

$$\begin{array}{ccc} X \times Y & \xrightarrow{\sigma_{X,Y}} & Y \times X \\ & \searrow^{id_X \otimes id_Y} & \downarrow \sigma_{Y,X} \\ & & X \times Y \end{array} \tag{B.6}$$

$$\begin{array}{ccc} X & \xrightarrow{\lambda_X} & \{*\} \times X \\ & \searrow^{\rho_X} & \downarrow \sigma_{\{*\},X} \\ & & X \times \{*\} \end{array} \tag{B.7}$$

Similarly to the triangle B.4, both these triangles commutes since both paths of the upper triangle gives

$$\{((x,y),(x,y)) \mid x \in X, y \in Y\},$$

while both paths of the lower triangle gives

$$\{(x,(x,\{*\})) \mid x \in X\}.$$

The hexagon

$$\begin{array}{ccccc} X \times (Y \times Z) & \xrightarrow{\alpha_{\_}} & (X \times Y) \times Z & \xrightarrow{\alpha_{(X\times Y),Z}} & Z \times (X \times Y) \\ \downarrow{\scriptstyle id_X \times \sigma_{Y,Z}} & & & & \downarrow{\scriptstyle \alpha_{\_}} \\ X \times (Z \times Y) & \xrightarrow[\alpha_{\_}]{} & (X \times Z) \times Y & \xrightarrow[\sigma_{X,Y} \times id_Z]{} & (Z \times X) \times Y \end{array} \tag{B.8}$$

given by equation 3.18, commutes as well. Indeed, in an analogous way to the coherence diagram B.2, both paths yields:

$$\{((x,(y,z)),((z,x),y)) \mid x \in X, y \in Y, z \in Z\}.$$

So **Set** and **Rel** are both symmetric monoidal categories as expected. The next step is to verify that **Rel** is compact. Recall from section 4.2 that compact structure involves units and counits such that the coherence diagrams 4.3 and 4.4 commutes. Since the objects in **Rel** are self-dual, the coherence diagram 4.3 becomes

$$\begin{array}{ccccc} X & \xrightarrow{\rho_X} & X \times \{*\} & \xrightarrow{id_X \times \eta_X} & X \times (X \times X) \\ \downarrow{\scriptstyle id_X} & & & & \downarrow{\scriptstyle \alpha_{X,X,X}} \\ X & \xleftarrow[\lambda_X^{-1}]{} & \{*\} \times X & \xleftarrow[\epsilon_X \times id_X]{} & (X \times X) \times X \end{array} \tag{B.9}$$

in which the unit and counit are given by

$$\eta_X : \{*\} \longrightarrow X \times X = \{(\{*\},(x,x)) \mid x \in X\}$$

and

$$\epsilon_X : X \times X \longrightarrow \{*\} = \{((x,x),\{*\}) \mid x \in X\},$$

respectively (see section 4.2.1). Indeed, with these identifications, the coherence condition B.9 commutes. Let us partition this verification into four parts:

(i) The composition

$$id_X \times \eta_X \circ \rho_X : X \longrightarrow X \times (X \times X),$$

is by definition a subset of

$$X \times (X \times (X \times X)),$$

which is given by

$$id_X \times \eta_X \circ \rho_X = \Big\{(x,(x',(x'',x'''))) \;\Big|\; \exists (x'''',\{*\}) \text{ s.t} \\ x\rho_X(x'''',\{*\}) \text{ and } (x'''',\{*\}) id_X \times \eta_X(x',(x'',x'''))\Big\}. \tag{B.10}$$

Now, by the definition of ρ_X we have that $x = x'''$ while the definitions of id_X and η_X entails that $x ='''' = x'$ and $x'' = x'''$ respectively. Thus,

$$id_X \times \eta_X \circ \rho_X := \{x, (x, (x', x'))) \mid x, x' \in X\}.$$

(ii) Hence, the composition

$$\alpha \circ ((id_X \times \eta_X) \circ \rho_X) : X \longrightarrow (X \times X) \times X,$$

which by definition is a subset of

$$X \times (X \times (X \times X)),$$

is given by

$$\alpha \circ ((id_X \times \eta_X) \circ \rho_X) = \{x, (x, x'), x') \mid x,x' \in X\}.$$

(iii) The composite

$$(\epsilon_X \times id_X) \circ (\alpha \circ ((id_X \times \eta_X) \circ \rho_X)),$$

is by definition a subset of

$$X \times (\{*\}, X),$$

which is given by

$$(\epsilon_X \times id_X) \circ (\alpha \circ ((id_X \times \eta_X) \circ \rho_X)) = \Big\{(x, (\{*\}, x')) \;\Big|\; \exists((x'', x'''), x'''') \text{ s.t.}$$
$$x\alpha \circ (id_X \times \eta_X) \circ \rho_X((x'', x'''), x'''') \text{ and } ((x'', x'''), x'''')\epsilon_X \times id_X(*, x'). \quad \text{(B.11)}$$

By the definition of ϵ_X we have that $x'' = x'''$ while the definition of id_X yields $x'''' = x'$. Moreover, form part (ii) we have $x = x''$ and $x''' = x''''$. All this entails that $x = x' = x'' = x''' = x''''$. Hence,

$$(\epsilon_X \times id_X) \circ (\alpha \circ ((id_X \times \eta_X) \circ \rho_X)) = \{(x, (*, x)) \mid x \in X\}.$$

(iv) Lastly we have the composition

$$\lambda_X^{-1} \circ (\epsilon_X \times id_X) \circ (\alpha \circ ((id_X \times \eta_X) \circ \rho_X)),$$

which by definition is a subset of

$$X \times X.$$

Thus, the final composition gives

$$\lambda_X^{-1} \circ (\epsilon_X \times id_X) \circ (\alpha \circ ((id_X \times \eta_X) \circ \rho_X)) = \{(x, x) \mid x \in X\},$$

which is the identity relation as required. The verification of the dual coherence diagram 4.4 proceeds analogously. So, **Rel** is indeed compact.

Lastly we verify that **Rel** is †-compact. Recall from section 4.2.1 that we identified the dagger with the relation converse. That is, for the relation of type $R : X \longrightarrow Y$ its converse $R^{\cup} : Y \longrightarrow X$ is defined as follows:

$$R^{\cup} := \{(y,x) \mid xRy\}.$$

We then defined the contravariant identity-on-objects involutive functor to have the following properties:

$$\dagger : |\mathbf{Rel}^{op}| \longrightarrow |\mathbf{Rel}| :: X \longmapsto X,$$
$$\dagger : \mathbf{Rel}^{op}(X,Y) \longrightarrow \mathbf{Rel}(Y,X) :: R \longmapsto R^{\cup}.$$

Now, from definition B.1 above we have that the Cartesian product of two relations is given by

$$(R_1 \times R_2)^{\dagger} = \{((y,y'),(x,x')) \mid xR_1y \text{ and } x'R_2y'\} = R_1^{\dagger} \times R_2^{\dagger},$$

In order to verify that these properties induces †-compact structure on **Rel**, we need to verify that (see section 4.2)

1. $\alpha^{\dagger} = \alpha^{-1}, \lambda^{\dagger} = \lambda^{-1}, \rho^{\dagger} = \rho^{-1}, \sigma^{\dagger} = \sigma^{-1}$, and,
2. $\sigma \circ \epsilon_X^{\dagger} = \epsilon_X^{\dagger} = \eta_X$.

The first requirement is trivial since the inverse off all these morphisms is the relation converse. The second requirement is assured if the following coherence diagram commutes:

$$\begin{array}{ccc} \{*\} & \xrightarrow{\epsilon_X^{\dagger}} & X \times X \\ & \searrow^{\eta_X} & \downarrow{\scriptstyle \sigma_{X,X}} \\ & & X \times X \end{array} \tag{B.12}$$

Since it follows from

$$\epsilon_X := \{((x,x),*) \mid x \in X\},$$

that

$$\epsilon_X^{\dagger} := \{(*,(x,x)) \mid x \in X\},$$

we have

$$\sigma \circ \epsilon_X^{\dagger} = \epsilon_X^{\dagger} = \eta_X,$$

making the coherence diagram commute.

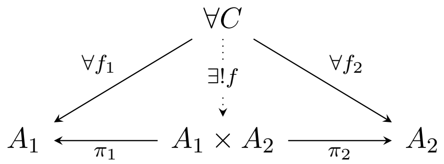
Thus, the category **Rel** is indeed a dagger compact category.

C

In this appendix we present a proof of the following proposition:

Proposition (Cartesian categories admits symmetric monoidal structure). *Given a Cartesian category $\boldsymbol{C}$, each choice of a product for each pair of objects always defines a symmetric monoidal structure on $\boldsymbol{C}$ with $A \otimes B := A \times B$, and where the terminal object $\top$ is identified with the monoidal unit* I.

Recall from section **??** that we could define a product through the following diagram



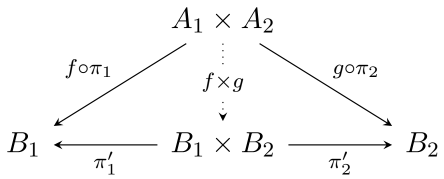
(C.1)

in which, for any object C in our category there exists a unique morphism $f : C \longrightarrow A_1 \times A_2$, such that this diagram commutes.

For Cartesian categories, all objects admits products. So, for morphisms $f : A_1 \longrightarrow B_1$ and $g : A_2 \longrightarrow B_2$ let

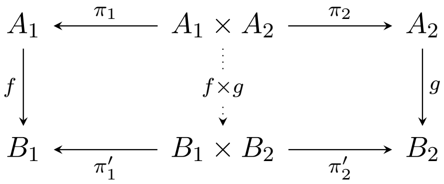
$$f \times g : A_1 \times A_2 \longrightarrow B_1 \times B_2$$

be the unique morphisms such that the following diagram commutes



(C.2)

It then immediately follows that the following diagram commutes as well



(C.3)

In section **??** we also gave an alternative way in which products could be defined, namely through the operations we called 'recombine' and 'decompose'. In particular, using those operations we have that for any morphisms f,

$$\langle \pi_1 \circ f, \pi_2 \circ f \rangle = f. \tag{C.4}$$

Indeed, for some Cartesian category $\mathbf{C}$, let the morphism $f \in \mathbf{C}$ be of type $f : A_1 \times A_2 \longrightarrow B_1 \times B_2$. We can then explicitly show that equation C.4 is valid by first applying the operation 'decompose' followed the operation 'recombine' on the homs-set $f \in \mathbf{C}(A_1 \times A_2, B_1 \times B_2)$:

$$Dec : \mathbf{C}(A_1 \times A_2, B_1 \times B_2) \longrightarrow \mathbf{C}(A_1 \times A_2, B_1) \times \mathbf{C}(A_1 \times A_2, B_2) :: f \longmapsto (\pi_1 \circ f, \pi_2 \circ f)$$

followed by

$$Rec : \mathbf{C}(A_1 \times A_2, B_1) \times \mathbf{C}(A_1 \times A_2, B_2) \longrightarrow \mathbf{C}(A_1 \times A_2, B_1 \times B_2) :: (\pi_1 \circ f, \pi_2 \circ f) \longmapsto \langle \pi_1 \circ f, \pi_2 \circ f \rangle.$$

Since we start and end up with the same hom-set $\mathbf{C}(A_1 \times A_2, B_1 \times B_2)$, we can indeed conclude that $f = \langle \pi_1 \circ f, \pi_2 \circ f \rangle$.

Using equation C.4 for $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : B \longrightarrow D$ we have

$$\begin{aligned} \langle g, h \rangle \circ f &= \langle \pi_1 \circ (\langle g, f \rangle \circ f), \pi_2 \circ (\langle g, h \rangle) && \text{(Equation C.4)} \\ &= \langle (\pi_1 \circ \langle g, h \rangle) \circ f, (\pi_2 \circ \langle g, h \rangle) \circ f \rangle && \text{(Associativity)} \\ &= \langle g \circ f, h \circ f \rangle. \end{aligned} \tag{C.5}$$

In the last equality we make use of 'decomposition'. More explicitly, since

$$Dec : \mathbf{C}(B, C \times D) \longrightarrow \mathbf{C}(B, C) \times \mathbf{C}(B, D) :: \langle g, h \rangle \longmapsto (\pi_1 \circ \langle g, h \rangle, \pi_2 \circ \langle g, h \rangle)$$

and $g \in \mathbf{C}(B, C)$, $h \in \mathbf{C}(C, D)$, we have that $g = \pi_1 \circ \langle g, h \rangle$ and $h = \pi_2 \circ \langle g, h \rangle$.

Using this result together with diagram C.2 for morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h : B \longrightarrow D$ and $k : B \longrightarrow E$ we have

$$\begin{aligned} (h \times k) \circ \langle f, g \rangle &= \langle h \circ \pi_1, k \circ \pi_2 \rangle' \circ \langle f, g \rangle && \text{(Diagram C.2)} \\ &= \langle h \circ \pi_1 \circ \langle f, g \rangle, k \circ \pi_2 \circ \langle f, g \rangle \rangle' && \text{(Equation C.5)} \\ &= \langle h \circ f, k \circ g \rangle', \end{aligned} \tag{C.6}$$

where $\langle -, - \rangle'$ is the 'recombine' operator relative to $(\pi_1' \circ -, \pi_2' \circ -)$. More explicitly, the first equality tells us that $(h \times k) = \langle h\pi_1, k \circ \pi_2 \rangle'$, which we can verify using the 'decomposition' and 'recombine' operations together with diagram C.2. Indeed, since $h \times k \in \mathbf{C}(B \times C, D \times E)$, we have

$$Dec : \mathbf{C}(B \times C, D \times E) \longrightarrow \mathbf{C}(B \times C, D) \times \mathbf{C}(B \times C, E) :: (h \times k) \longmapsto (h \times k \circ \pi_1', h \times k \circ \pi_2').$$

Making use of diagram C.2 we have $h \times k \circ \pi_1' = h \circ \pi_1$ and $h \times k \circ \pi_2' = k \circ \pi_2$ so

$$Rec : \mathbf{C}(B \times C, D) \times \mathbf{C}(B \times C, E) \longrightarrow \mathbf{C}(B \times C, D \times E) :: (h \times \pi_1, k \circ \pi_2) \longmapsto \langle h \circ \pi_1, k \circ \pi_2 \rangle',$$

from which we have $(h \times k) = \langle h \circ \pi_1, k \circ \pi_2 \rangle'$.

The last equality in C.6 states that $f = \pi_1 \circ \langle f, g \rangle$ and $g = \pi_2 \circ \langle f, g \rangle$. Indeed, since $f \in \mathbf{C}(A,B)$ and $g \in \mathbf{C}(A,C)$ we have

$$Rec : \mathbf{C}(A,B) \times \mathbf{C}(A,C) \longrightarrow \mathbf{C}(A, B \times C) :: (f,g) \longmapsto \langle f, g \rangle,$$

so,

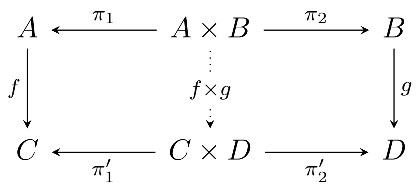
$$Dec : \mathbf{C}(A, B \times C) \longrightarrow \mathbf{C}(A,B) \times \mathbf{C}(A,C) :: \langle f, g \rangle \longmapsto (\pi_1 \circ \langle f, g \rangle, \pi_2 \circ \langle f, g \rangle),$$

from which we have $f = \pi_1 \circ \langle f, g \rangle$ and $g = \pi_2 \circ \langle f, g \rangle$.

Using the results C.5 and C.6, we are now in a position to verify that $- \times -$ is bifunctorial. That is, we now verify that

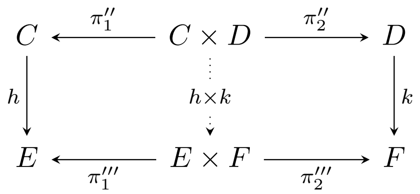
$$(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g),$$

for any morphisms $f : A \longrightarrow C$, $g : B \longrightarrow D$, $h : C \longrightarrow E$ and $k : D \longrightarrow F$. In order to do so, we will make use diagram C.3 which for $f \times g$ becomes



(C.7)

while for $h \times k$ we have



(C.8)

Now, from these diagrams together with equations C.5 and C.6 we have

$$\begin{aligned} (h \times k) \circ (f \times g) &= \langle h \circ \pi_1'', k \circ \pi_2'' \rangle''' \circ \langle f \circ \pi_1, g \circ \pi_2 \rangle' \\ &= \langle h \circ \pi_1'' \circ \langle f \circ \pi_1, g \circ \pi_2 \rangle', k \circ \pi_2'' \circ \langle f \circ \pi_1, g \circ \pi_2 \rangle' \rangle''' \\ &= \langle h \circ f \circ \pi_1, k \circ g \circ \pi_2 \rangle''' \\ &= (f \circ h) \times (g \circ k). \end{aligned} \tag{C.9}$$

With bifunctorality now established, the reminder of this proof will be concerned with constructing the required natural isomorphisms. We will only focus on the required naturality conditions and leave the coherence diagrams to the reader.

Let $!_A$ be the unique morphism of type $A \longrightarrow \top$. We can then identify the left unit natural isomorphism by the following morphism:

$$\lambda_A := \langle !_A, id_A \rangle : A \longrightarrow \top \times A.$$

Indeed, using this definition, the defining naturality condition for the left unit, 3.12, becomes

$$
\begin{array}{ccc}
A & \xrightarrow{\lambda_A} & \top \times A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_\top \times f} \\
B & \xrightarrow[\lambda_B]{} & \top \times B
\end{array}
\tag{C.10}
$$

which commutes since

$$
\begin{aligned}
\langle !_B, id_B \rangle \circ f &= \langle !_B \circ f, id_B \circ f \rangle \qquad \text{(Equation C.5)} \\
&= \langle !_A, f \circ id_A \rangle \\
&= (!_\top \times f) \circ \langle !_A, id_A \rangle .
\end{aligned}
\tag{C.11}
$$

That is, from the commutation of C.10 we have established that λ is natural. In fact, λ is moreover an isomorphism with π_2 as its inverse. In order to verify this we must have that

1. $\pi_2 \circ \lambda_A = id_A$ and,
2. $\lambda_A \circ \pi_2 = id_{\top \times A}$.

The fact that $\pi_2 \circ \lambda_A = id_A$, holds by definition, while $\lambda_A \circ \pi_2 = id_{\top \times A}$ can be verified from

$$
\begin{array}{ccccc}
\top & \xleftarrow{\pi_1} & \top \times A & \xrightarrow{\pi_2} & A \\
{\scriptstyle !_\top}\downarrow & & \vdots\, {\scriptstyle !_\top \times id_A} & & \downarrow{\scriptstyle id_A} \\
\top & \xleftarrow[\pi_1']{} & \top \times A & \xrightarrow[\pi_2']{} & A
\end{array}
\tag{C.12}
$$

and the fact that $\top$ is terminal from which we have

$$!_{\top \times A} = !_\top \circ \pi_1 = !_A \circ \pi_2$$

Thus, the following diagram commutes:

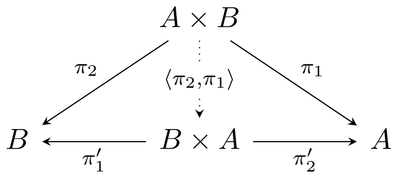
$$
\begin{array}{ccccc}
 & & \top \times A & & \\
 & \swarrow {\scriptstyle !_A \circ \pi_2} & \vdots\, {\scriptstyle \langle !_A \circ \pi_2, id_A \circ \pi_2 \rangle} & {\scriptstyle id_A \circ \pi_2} \searrow & \\
\top & \xleftarrow[\pi_1']{} & \top \times A & \xrightarrow[\pi_2']{} & A
\end{array}
\tag{C.13}
$$

By uniqueness it then follows that $\langle !_A \circ \pi_2, id_A \circ \pi_2 \rangle = !_\top \times id_A$, and hence

$$\lambda_A \circ \pi_2 := \langle !_A, id_A \rangle \circ \pi_2 = \langle !_A \circ \pi_2, id_A \circ \pi_2 \rangle = !_\top \times id_A = id_{\top \times A}.$$

By an analogous process we can identify the right unit isomorphism by $\rho := \langle id_A !_A \rangle$.

The symmetry natural isomorphism $\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$ defined thorough the naturality diagram 3.13, is identified through the following commuting diagram:

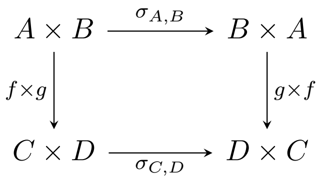


(C.14)

That is, the symmetry natural isomorphism is defined by the following morphism

$$\sigma_{A,B} := \langle \pi_2, \pi_1 \rangle : A \times B \longrightarrow B \times A.$$

Using $\sigma_{A,B} := \langle \pi_2, \pi_1 \rangle$ we have that the defining naturality diagram for symmetry, 3.13, becomes

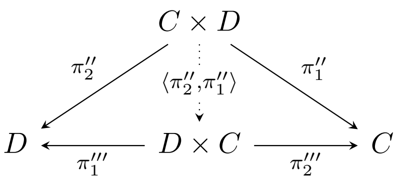


(C.15)

which commutes since

$$\begin{aligned} \langle \pi_2, \pi_1 \rangle \circ (f \times g) &= \langle \pi_2 \circ (f \times g), \pi_1 \circ (f \times g) \rangle'' \quad &\text{(Equation C.5)} \\ &= \langle g \circ \pi_2, f \circ \pi_1 \rangle \\ &= (g \times f) \circ \langle \pi_2, \pi_1 \rangle, &\text{(Equation C.6)} \end{aligned} \tag{C.16}$$

where $\langle -, - \rangle''$ is the 'recombine' operator relative to $(\pi_1'' \circ -, \pi_2'' \circ -)$ which resides within



(C.17)

making σ natural. Now, relying on uniqueness of the symmetry morphism in C.14 we have that σ is a natural isomorphism with $\langle \pi_2', \pi_1' \rangle$ as its inverse since

1. $\langle \pi_2', \pi_1' \rangle \circ \langle \pi_2, \pi_1 \rangle = id_{A \times B}$ and,

2. $\langle \pi_2, \pi_1 \rangle \circ \langle \pi_2', \pi_1' \rangle = id_{B \times A}$.

Lastly, in order to construct the associativity natural isomorphism, let us define the following notation for the relevant projections:

$$A \xleftarrow{\pi_1} A \times (B \times C) \xrightarrow{\pi_2} B \times C \qquad \text{and} \qquad B \xleftarrow{\pi_1'} B \times C \xrightarrow{\pi_2'} C.$$

We then define a morphism of type $A \times (B \times C) \longrightarrow A \times B$ through the following commuting diagram:

$$\begin{array}{ccccc} & & A\times(B\times C) & & \\ & \swarrow^{\pi_1} & \vdots\ \langle\pi_1,\pi_1'\circ\pi_2\rangle & \searrow^{\pi_1'\circ\pi_2} & \\ A & \xleftarrow[\pi_1']{} & A\times B & \xrightarrow[\pi_2'']{} & B \end{array} \tag{C.18}$$

The associativity natural isomorphism $\alpha_{A,B,C}$, is then defined through

$$\begin{array}{ccccc} & & A\times(B\times C) & & \\ & \swarrow^{\langle\pi_1,\pi_1'\circ\pi_2\rangle} & \vdots\ \langle\langle\pi_1,\pi_1'\circ\pi_2\rangle,\pi_2'\circ\pi_2\rangle & \searrow^{\pi_2'\circ\pi_2} & \\ A\times B & \xleftarrow[\pi_1''']{} & (A\times B)\times C & \xrightarrow[\pi_2''']{} & C \end{array} \tag{C.19}$$

That is, the associativity natural isomorphism is defined by the following morphism

$$\alpha_{A,B,C} := \langle\langle\pi_1, \pi_1' \circ \pi_2\rangle, \pi_2' \circ \pi_2\rangle : A \times (B \times C) \longrightarrow (A \times B) \times C.$$

Similarly as in the case of symmetry, using $\alpha_{A,B,C} := \langle\langle\pi_1, \pi_1' \circ \pi_2\rangle, \pi_2' \circ \pi_2\rangle$ we have that the defining naturality diagram for associativity, 3.11, becomes

$$\begin{array}{ccc} A\times(B\times C) & \xrightarrow{\alpha_{A,B,C}} & (A\times B)\times C \\ {\scriptstyle f\times(g\times h)}\downarrow & & \downarrow{\scriptstyle (f\times g)\times h} \\ A'\times(B'\times C') & \xrightarrow[\alpha_{A',B',C'}]{} & (A'\times B')\times C' \end{array} \tag{C.20}$$

which commutes since

$$\begin{aligned} \langle\langle\pi_1, \pi_1' \circ \pi_2\rangle, \pi_2' \circ \pi_2\rangle \circ f \times (g \times h) &= \langle\langle\pi_1, \pi_1' \circ \pi_2\rangle \circ f \times (g \times h), (\pi_2' \circ \pi_2) \circ f \times (g \times h)\rangle'' \\ &= \langle f \times g \circ \langle\pi_1, \pi_1' \circ \pi_2\rangle, h \circ (\pi_2' \circ \pi_2)\rangle \\ &= (f \times g) \times h \circ \langle\langle\pi_1, \pi_1' \circ \pi_2\rangle, \pi_2' \circ \pi_2\rangle, \end{aligned} \tag{C.21}$$

making α natural. Relying on uniqueness of the associativity morphism in C.19 we have that α is a natural isomorphism with $\langle\pi_1''', \pi_2'''\rangle$ as its inverse since

1. $\langle\pi_1''', \pi_2'''\rangle \circ \langle\langle\pi_1, \pi_1' \circ \pi_2\rangle, \pi_2' \circ \pi_2\rangle = id_{A\times(B\times C)}$ and,
2. $\langle\langle\pi_1, \pi_1' \circ \pi_2\rangle, \pi_2' \circ \pi_2\rangle \circ \langle\pi_1''', \pi_2'''\rangle = id_{(A\times B)\times C}$.

So, given a Cartesian category **C**, we have shown that we can endow symmetric monoidal structure to it. Indeed, bifunctorality is identified through equation C.9, the left unit natural isomorphism is identified through commutation of diagram C.10, the symmetry natural isomorphism is identified through commutation of diagram C.15 and associativity is identified through commutation of diagram C.19.

D

In this appendix we present a proof of the following proposition:

Proposition. *All Cartesian categories admits a uniform copying operation.*

First, recall from section 4.4 that a uniform copying relation, in categorical terms, is given by a diagonal which we defined through the commutation of the following diagram:

$$\begin{CD} A @>f>> B \\ @V\Delta_AVV @VV\Delta_BV \\ A\otimes A @>>f\otimes f> B\otimes B \end{CD} \tag{D.1}$$

Now, in order to verify that this diagram commutes for arbitrary Cartesian categories, let the diagonal be given by

$$\Delta_A := \langle id_A, id_A\rangle : A \longrightarrow A \otimes A,$$

and let $f : A \longrightarrow B$ be a arbitrary morphism. Using that any Cartesian category can be endowed with symmetric monoidal structure (see appendix C), diagram D.1 becomes

$$\begin{CD} A @>f>> B \\ @V\Delta_AVV @VV\Delta_BV \\ A\times A @>>f\times f> B\times B \end{CD} \tag{D.2}$$

while the diagonal becomes

$$\Delta_A := \langle id_A, id_A\rangle : A \longrightarrow A \times A.$$

Furthermore, since we now consider Cartesian categories endowed with symmetric monoidal structure, recall that for morphisms of appropriate type we have that $\langle g, h\rangle = \langle g \circ f, h \circ g\rangle$ and $(h \times k) \circ \langle f, g\rangle = \langle h \circ f, k \circ g\rangle'$ (see equations C.5 and C.6). Making use of these equations we have

$$\langle id_B, id_B\rangle \circ f = \langle id_B \circ f, id_B \circ f\rangle = \langle f \circ id_A, f \circ id_A\rangle = (f \times f) \circ \langle id_A, id_A\rangle,$$

making diagram D.2 commute. So the diagonal Δ is indeed a natural transformation and hence a uniform copying operation.