# Entanglement and its Quantification


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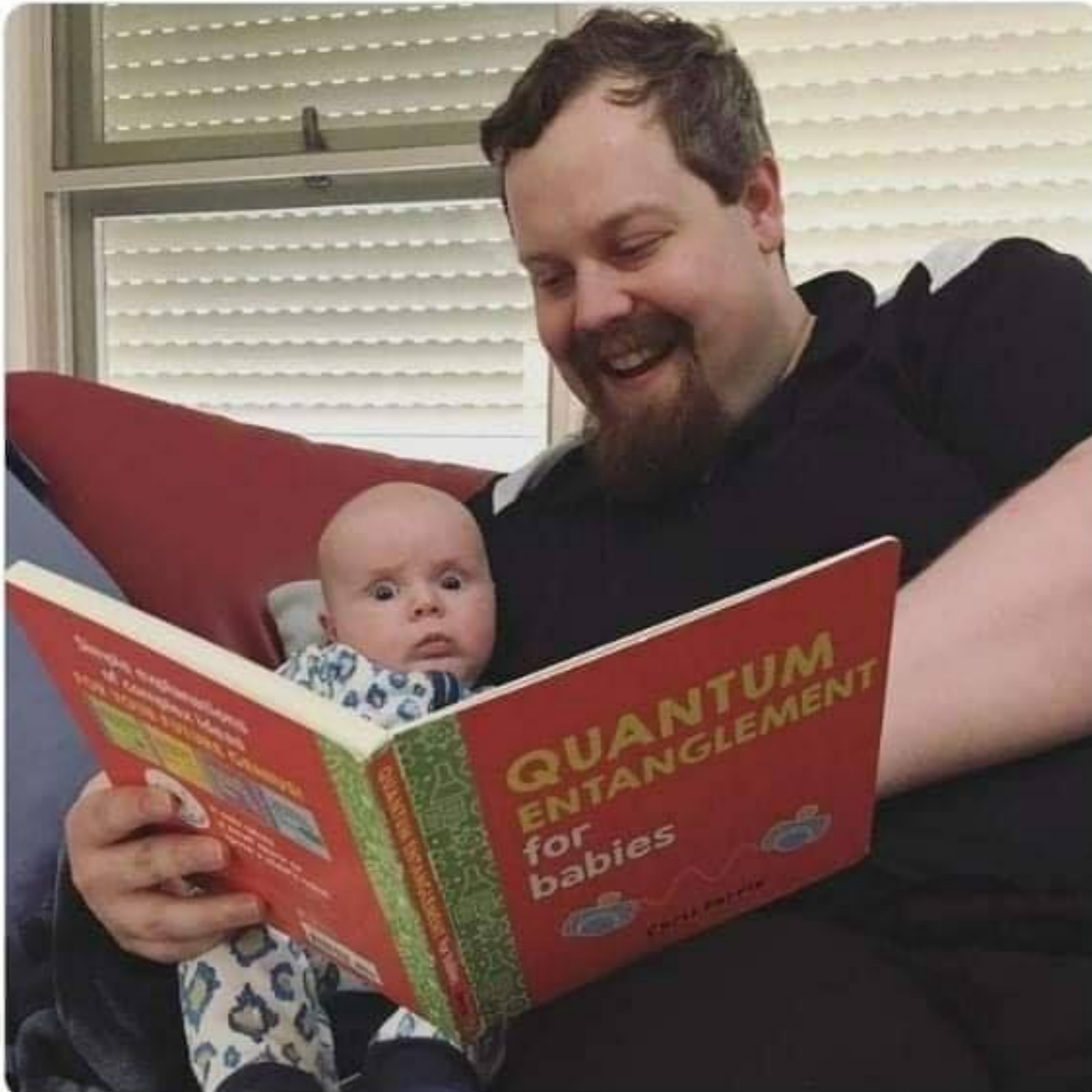

QUANTUM ENTANGLEMENT
for babies

# Content of the talk (first part)

**Basic concepts of quantum information**

- Pure states
- Mixed states

**Quantum operations**

- LOCC

**Pure state entanglement**

- Von Neumann entropy

# Basics of quantum mechanics/information

# Pure States

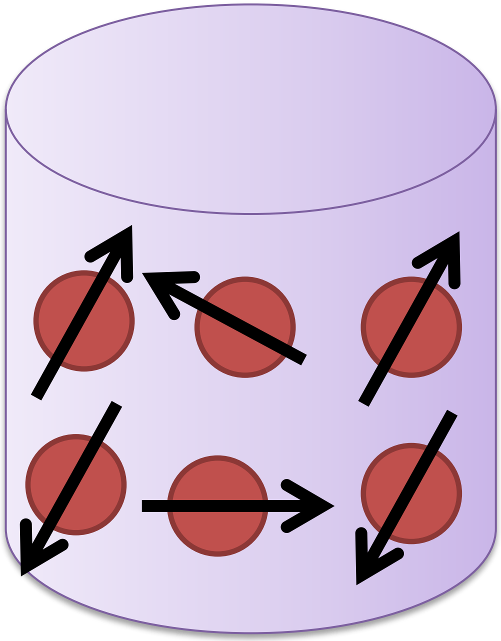
**Systems that are fully explained by a unique wave function** $|\psi\rangle$

**Example 1: A qubit** $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

**Example 2: Bipartite systems**

$$|\psi\rangle_{AB} = \frac{|00\rangle + |01\rangle}{\sqrt{2}} = |0\rangle_A \otimes (\frac{|0\rangle + |1\rangle}{\sqrt{2}})_B$$

# Mixed States



**Ensemble of pure states:**

$$\left\{ \begin{array}{ll} P_1 : & |\psi_1\rangle \\ P_2 : & |\psi_2\rangle \\ \vdots & \\ P_n : & |\psi_n\rangle \end{array} \right.$$

**Density matrix for explaining the state of particles in the box:**

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

# Properties of the Density Matrix

1. **Hermiticity:** $\rho = \rho^{+}$

2. **Trace one:** $Tr(\rho) = 1$

3. **Positivity:** $\rho \geq 0$

**What positivity means:** $\forall |\psi\rangle : \quad \langle\psi|\rho|\psi\rangle \geq 0$

**In particular for** $|\psi\rangle = |\lambda_i\rangle : \quad \rho|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$

$$\langle\psi|\rho|\psi\rangle = \langle\lambda_i|\rho|\lambda_i\rangle = \lambda_i \geq 0$$

**All eigenvalues of a density matrix should be equal or greater than zero**

# Decomposition of the Density Matrix

**Decomposition is not unique**

$$\rho = \sum_i p_i \left|\psi_i\right\rangle\left\langle\psi_i\right| = \sum_j q_j \left|\phi_j\right\rangle\left\langle\phi_j\right|$$

**Note that the set of quantum states in each decomposition are not necessarily orthogonal unless for the eigenvectors**

**Example:** $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$

$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**where** $|+\rangle = \frac{|0\rangle + |1\rangle}{2}$ **&** $|-\rangle = \frac{|0\rangle - |1\rangle}{2}$

# Purity

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_j q_j |\lambda_j\rangle\langle\lambda_j|$$

$$\rho^2 = \sum_j q_j^2 |\lambda_j\rangle\langle\lambda_j| \quad \Rightarrow \quad Tr(\rho^2) = \sum_j q_j^2 \le 1$$

**Purity:** $P = Tr(\rho^2)$ $\qquad \frac{1}{d} \le P \le 1$

**Pure states:** $\rho = |\psi\rangle\langle\psi| \Rightarrow P = 1$

**Maximally mixed state:** $\rho = \frac{I}{d} \Rightarrow P = \frac{1}{d}$

# Von Neumann Entropy

$$\rho = \sum_j \lambda_j \left|\lambda_j\right\rangle\left\langle\lambda_j\right| \qquad \left\langle\lambda_i\middle|\lambda_j\right\rangle = \delta_{ij}$$

Eigenstates

Von Neumann Entropy: $S(\rho) = -Tr\left(\rho\log(\rho)\right) = -\sum_j \lambda_i \log(\lambda_i)$

Pure states: $\rho = \left|\psi\right\rangle\left\langle\psi\right| \Rightarrow S(\rho) = 0$

Note that : 0 log(0)=0

Maximally mixed state: $\rho = \frac{I}{d} \Rightarrow S(\rho) = \log(d)$

Both purity and von Neumann entropy quantify the purity (or equivalently the mixedness) of the system

# Evolution

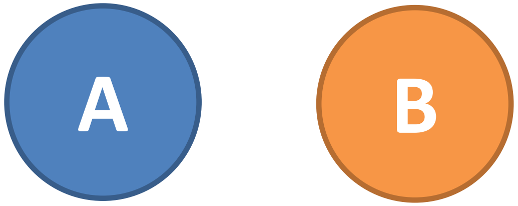
**Closed systems:** $\rho \rightarrow U\rho U^{+}, \quad UU^{+} = U^{+}U = I$

**Open systems:** $\rho \rightarrow \sum_k L_k \, \rho \, L_k^{+}, \quad \sum_k L_k^{+} L_k = I$

$\{L_k\}$ **are called Kraus operators**

# Concept of entanglement (pure states)

# Bipartite Systems




**Separable pure states:** $|\psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B$

$$|\psi\rangle_{AB} = \frac{|00\rangle + |01\rangle}{\sqrt{2}} = |0\rangle_A \otimes (\frac{|0\rangle + |1\rangle}{\sqrt{2}})_B$$

**Non-separable states are called entangled states**

$$|\psi\rangle_{AB} \neq |\alpha\rangle_A \otimes |\beta\rangle_B$$

$$|\psi\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}$$

# Schmidt Decomposition

**The most general state:** $|\psi\rangle_{AB} = \sum_{i,j} \alpha_{ij} |i_A, j_B\rangle$

$$\langle i_A | i'_A \rangle = \delta_{ii'}, \qquad \langle j_B | j'_B \rangle = \delta_{jj'}$$

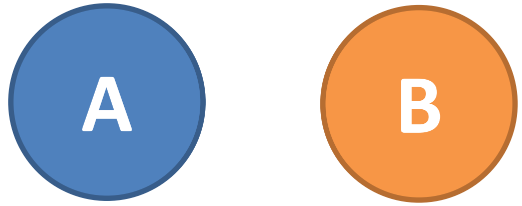
**Schmidt basis:**

$$|\psi\rangle_{AB} = \sum_{i,j} \alpha_{ij} |i_A, j_B\rangle_{AB} = \sum_i \lambda_i |\tilde{i}_A, \tilde{i}_B\rangle$$

**Properties of Schmidt decomposition**

- $\langle \tilde{i}_A | \tilde{i}_A' \rangle = \delta_{ii'}, \qquad \langle \tilde{i}_B | \tilde{i}_B' \rangle = \delta_{ii'}$
- $\lambda_i$'s are real and positive (Schmidt coefficients)

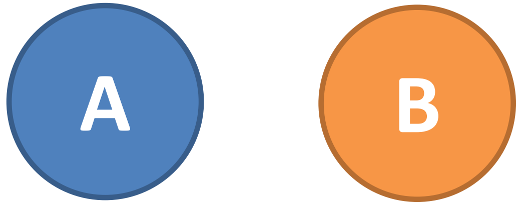
# State of the Subsystem




$$\rho_A = Tr_B(\rho_{AB}) = \sum_i \langle i_B | \rho_{AB} | i_B \rangle$$

$$\rho_B = Tr_A(\rho_{AB}) = \sum_i \langle i_A | \rho_{AB} | i_A \rangle$$

# Von Neumann Entropy




**Schmidt decomposition:** $|\psi\rangle_{AB} = \sum_i \lambda_i |\tilde{i}_A, \tilde{i}_B\rangle$

$$\langle \tilde{i}_A | \tilde{i}_A' \rangle = \delta_{ii'}, \quad \langle \tilde{i}_B | \tilde{i}_B' \rangle = \delta_{ii'}$$

**Subsystems:**

$$\rho_B = Tr_A(|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_i \lambda_i^2 |\tilde{i}_B\rangle\langle\tilde{i}_B|$$

$$\rho_A = Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_i \lambda_i^2 |\tilde{i}_A\rangle\langle\tilde{i}_A|$$

**If AB is pure:**

- **Purity:** $P_A = P_B$
- **Von Neumann Entropy:** $S(\rho_A) = S(\rho_B)$

# Separable Pure States

**Separable state:** $|\psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B$

**Subsystems:**
$$\begin{cases} \rho_A = |\alpha\rangle\langle\alpha| \\ \rho_B = |\beta\rangle\langle\beta| \end{cases} \Rightarrow \begin{cases} P_A = P_B = 1 \\ S(\rho_A) = S(\rho_B) = 0 \end{cases}$$

**In separable pure states the subsystems are also pure**

# Entangled Pure States

**Entangled states:** $|\psi\rangle_{AB} \neq |\alpha\rangle_A \otimes |\beta\rangle_B$

**Subsystems are not pure**

$$\begin{cases} \rho_A \neq |\alpha\rangle\langle\alpha| \\ \rho_B \neq |\beta\rangle\langle\beta| \end{cases} \Rightarrow \begin{cases} P_A = P_B < 1 \\ S(\rho_A) = S(\rho_B) > 0 \end{cases}$$

**Von-Neumann Entropy of the subsystem quantifies the entanglement**

# Example 1

**Maximally entangled states:**

$$|\psi\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}} \quad \Rightarrow \quad \rho_A = \frac{I}{2}, \quad \rho_B = \frac{I}{2}$$

$$\Rightarrow \begin{cases} P_A = P_B = \dfrac{1}{2} \\ S(\rho_A) = S(\rho_B) = \log(2) = 1 \end{cases}$$

**Maximally entangled states**

1. Subsystems are maximally mixed
2. The entropy of subsystems are maximal
3. The purity of the subsystems are minimal

# Example 2

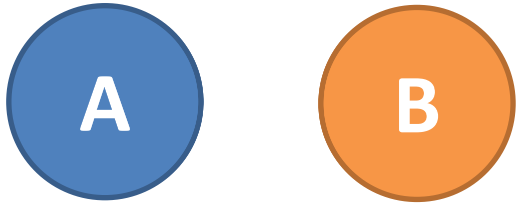
## Non-maximal entangled states:

$$|\psi\rangle_{AB} = \sqrt{\frac{1}{3}}|00\rangle_{AB} + \sqrt{\frac{2}{3}}|11\rangle_{AB} \Rightarrow \rho_A = \rho_B = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$$

$$\Rightarrow \begin{cases} P_A = P_B = \dfrac{5}{9} \\ S(\rho_A) = S(\rho_B) = -\dfrac{1}{3}\log(\dfrac{1}{3}) - \dfrac{2}{3}\log(\dfrac{2}{3}) \approx 0.9183 \end{cases}$$

**Entropy of the subsystem can quantify the amount of entanglement**

# Entanglement of Pure States




**Overall state:** $|\psi\rangle_{AB}$ → $\begin{cases} \rho_A = Tr_B(\rho_{AB}) \\ \rho_B = Tr_A(\rho_{AB}) \end{cases}$

**Entanglement between the two subsystems:** $E = S(\rho_A) = S(\rho_B)$

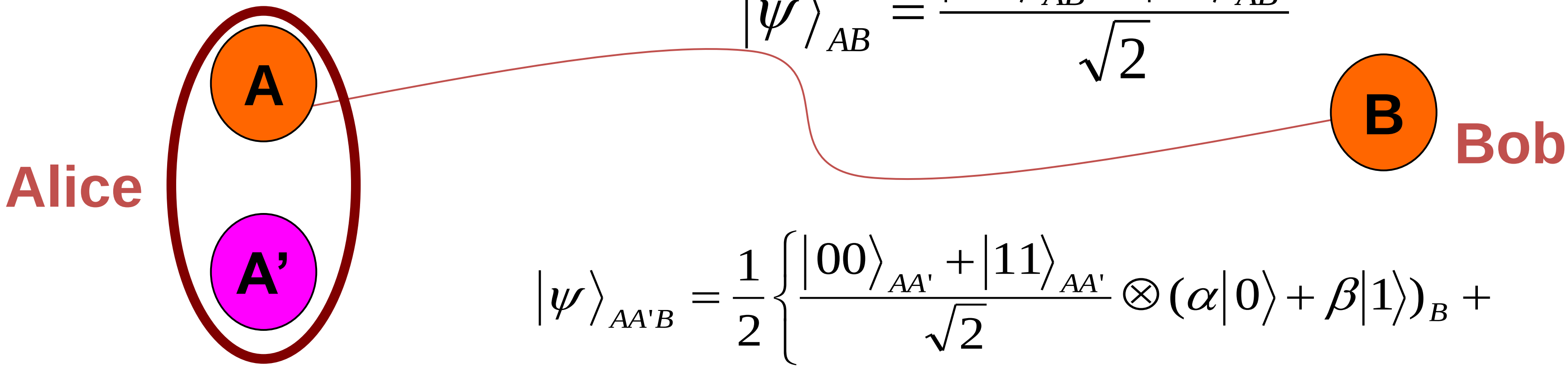
$$0 \leq E \leq \log(d)$$

**Separable states** — **Maximally entangled states**

**All entanglement measures are monotonic functions with respect to the von Neumann entropy**

# Application 1: Teleportation

$$|\psi\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}$$




$$|\psi\rangle_{A'} = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle_{AA'B} = \frac{1}{2}\left\{\frac{|00\rangle_{AA'} + |11\rangle_{AA'}}{\sqrt{2}} \otimes (\alpha|0\rangle + \beta|1\rangle)_B + \right.$$

$$\frac{|01\rangle_{AA'} + |10\rangle_{AA'}}{\sqrt{2}} \otimes \sigma_x(\alpha|0\rangle + \beta|1\rangle)_B +$$

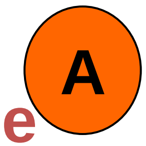
$$\frac{|01\rangle_{AA'} - |10\rangle_{AA'}}{\sqrt{2}} \otimes \sigma_y(\alpha|0\rangle + \beta|1\rangle)_B +$$

$$\left.\frac{|00\rangle_{AA'} - |11\rangle_{AA'}}{\sqrt{2}} \otimes \sigma_z(\alpha|0\rangle + \beta|1\rangle)_B\right\}$$

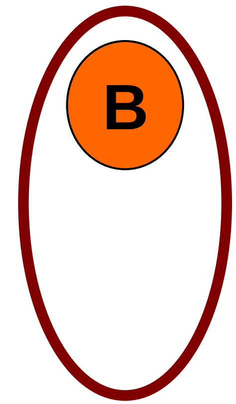
**With 2 classical bits of information Bob gets the exact state of Alice.**

# Application 2: Dense coding

Alice




$$|\psi\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}$$




Bob

$$I \otimes I|\psi\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}$$

$$\sigma_x \otimes I|\psi\rangle_{AB} = \frac{|01\rangle_{AB} + |10\rangle_{AB}}{\sqrt{2}}$$

$$\sigma_y \otimes I|\psi\rangle_{AB} = \frac{|01\rangle_{AB} - |10\rangle_{AB}}{\sqrt{2}}$$

$$\sigma_z \otimes I|\psi\rangle_{AB} = \frac{|00\rangle_{AB} - |11\rangle_{AB}}{\sqrt{2}}$$

**A single physical 2-level object carries two classical bits of information**

# Content of the talk (second part)

**Mixed state entanglement**

- **Operational measures:**
  1. **Entanglement cost**
  2. **Entanglement distillation**

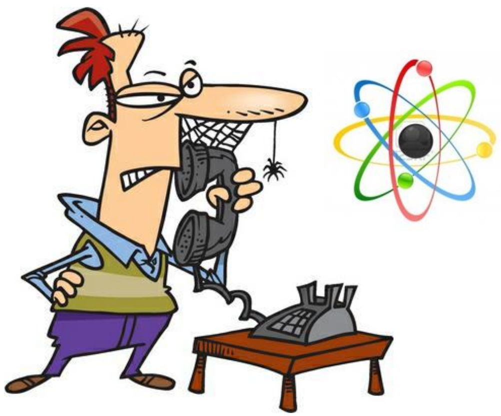
- **Asymptotic measures:**
  1. **Distance measures**
  2. **Entanglement formation**
  3. **Negativity**

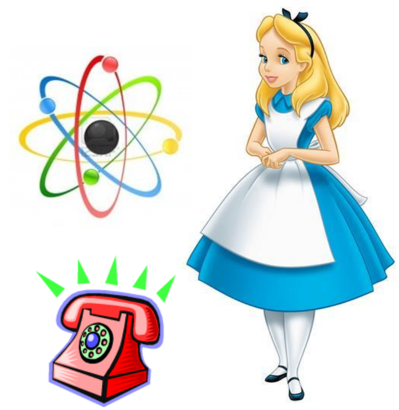
# Concept of entanglement (mixed states)

# Separable Mixed States

**Separable states:** $$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B$$

$$p_i \geq 0, \quad \sum_i p_i = 1$$





**With local operations and classical communications Alice and Bob can produce these kind of states**

# Examples for Separable States

$$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B$$

**Example 1 (Pure states):**

$$|\psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B \Rightarrow \rho_{AB} = |\alpha_A\rangle\langle\alpha_A| \otimes |\beta_A\rangle\langle\beta_A|$$

**Example 2:** $\rho_{AB} = \frac{1}{3}|0\rangle\langle 0| \otimes |+\rangle\langle +| + \frac{2}{3}|-\rangle\langle -| \otimes |1\rangle\langle 1|$

**Example 3:** $\rho_{AB} = \frac{1}{6} I \otimes |0\rangle\langle 0| + \frac{2}{6}|+\rangle\langle +| \otimes (I + \sigma_Z)$

# Entropy of the subsystem

**Separable states:** $$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B$$

**Subsystems:**
$$\begin{cases} \rho_B = Tr_A(\rho_{AB}) = \sum_i p_i \rho_i^B \\ \rho_A = Tr_B(\rho_{AB}) = \sum_i p_i \rho_i^A \end{cases}$$

**System is not entangled but:** $S(\rho_A)\ \text{or}\ S(\rho_B) > 0$

**Von Neumann entropy does not quantify the entanglement anymore as it originates from the initial entropy of the whole system**

# Examples

**Example 1 (Pure states):**

$$\rho_{AB} = |\alpha_A\rangle\langle\alpha_A| \otimes |\beta_A\rangle\langle\beta_A| \Rightarrow S(\rho_A) = S(\rho_B) = 0$$

**Example 2:**

$$\rho_{AB} = \frac{1}{6} I \otimes |0\rangle\langle 0| + \frac{2}{3} |+\rangle\langle +| \otimes |1\rangle\langle 1|$$

**Subsystems:**

$$\begin{cases} \rho_A = \frac{1}{6} I + \frac{2}{3} |+\rangle\langle +| \\ \rho_B = \frac{1}{3} |0\rangle\langle 0| + \frac{2}{3} |1\rangle\langle 1| \end{cases} \Rightarrow \begin{cases} S(\rho_A) = 0.6500 \\ S(\rho_B) = 0.9183 \end{cases}$$

$$S(\rho_A) \neq S(\rho_B)$$

# Entanglement of Mixed States

**Entangled states:** $$\rho_{AB} \neq \sum_i p_i \rho_i^A \otimes \rho_i^B$$

**How to quantify entanglement for a general mixed state?**

**There is not a unique entanglement measure**

**Classical actions:**

**Local Operations and Classical Communications (LOCC)**

# LOCC Maps

**Local Operations and Classical Communication (LOCC)**

1. Unitary evolution
2. Measurement

Informing the other party about the outcomes of measurements

# LOCC Kraus Operators

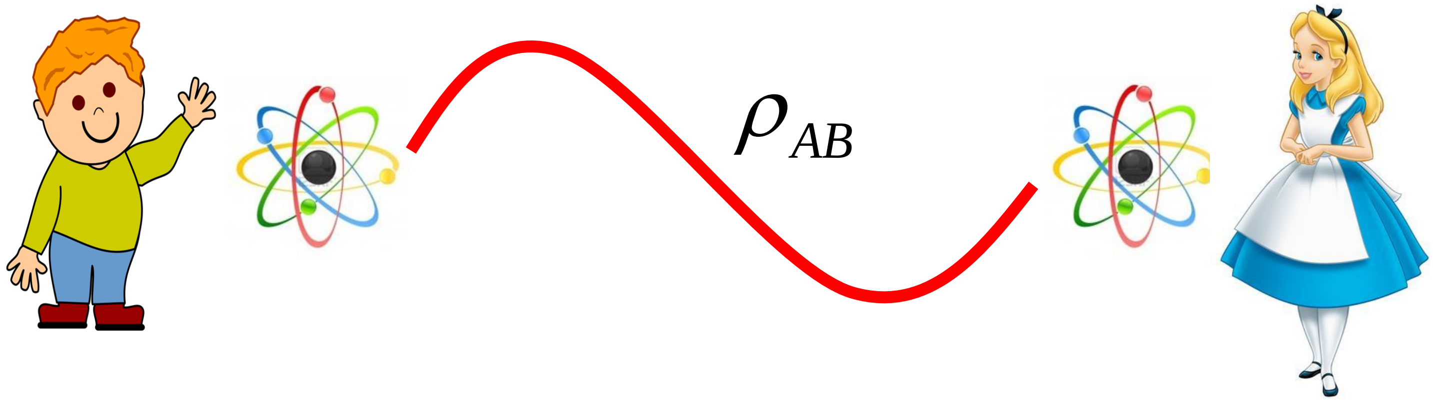
**General quantum operation:** $\rho \to \sum_k L_k \, \rho \, L_k^+, \quad \sum_k L_k^+ L_k = I$

**LOCC Kraus operators:** $L_K = A_K \otimes B_K$

$$\rho_{AB}^{final} = \sum_k A_k \otimes B_k \, \rho_{AB}^{\text{init}} \, A_k^+ \otimes B_k^+, \qquad \sum_k A_k^+ A_k \otimes B_k^+ B_k = I$$

**Trace preserving is equivalent to conservation of probabilities**

# Basic Properties for Entanglement Measures




1. $$E(\rho_{AB}) \in R^+$$

2. $$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B \Rightarrow E(\rho_{AB}) = 0$$

3. $$|\psi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_i |i_A, i_B\rangle \Rightarrow E(\rho_{AB}) \text{ is maximum}$$

4. $$\sigma_{AB} = \sum_k A_k \otimes B_k \; \rho_{AB} \; A_k^+ \otimes B_k^+ \Rightarrow E(\rho_{AB}) \geq E(\sigma_{AB})$$

# Extra Properties

5 **Convexity:**

$$E(\sum_i p_i \rho_i^{AB}) \leq \sum_i p_i E(\rho_i^{AB})$$

**Example:**

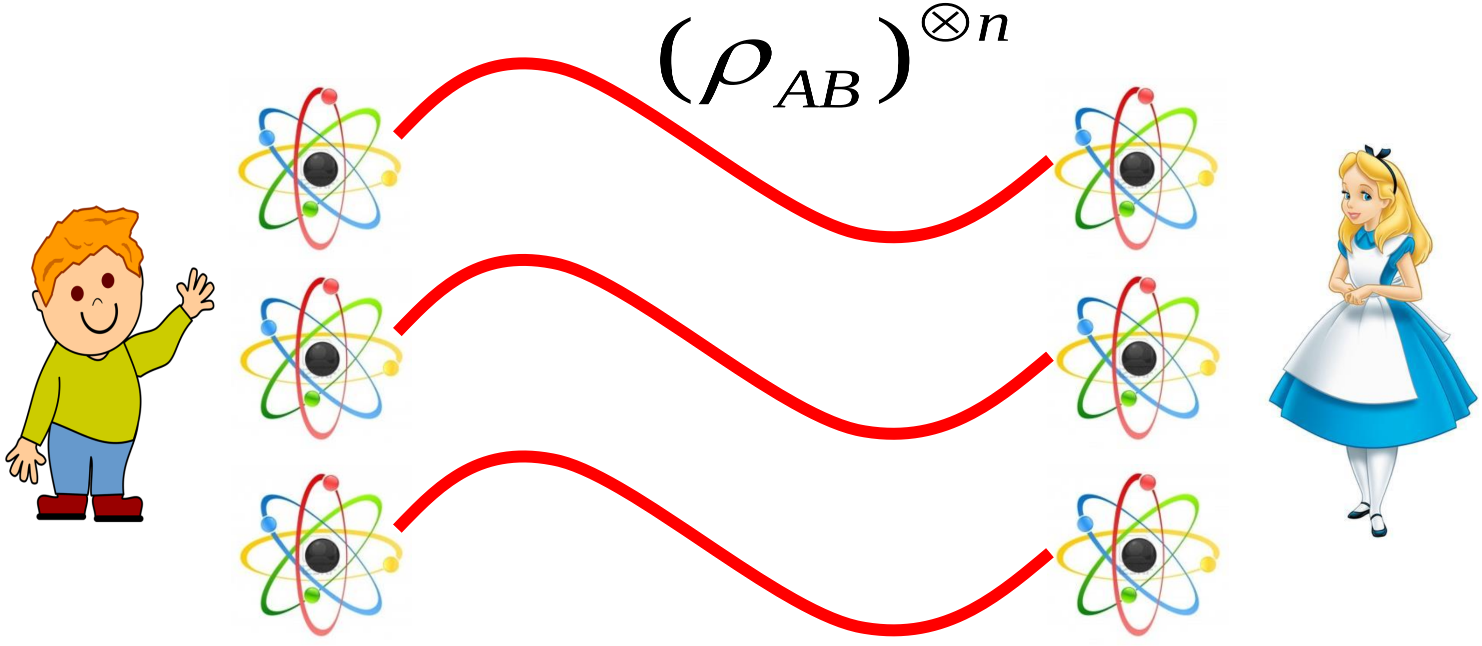
$$|\psi_1\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad |\psi_2\rangle_{AB} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$\rho_{AB} = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$$

# Extra Properties

6 **Additivity:** $E(\rho^{\otimes n}) = nE(\rho)$

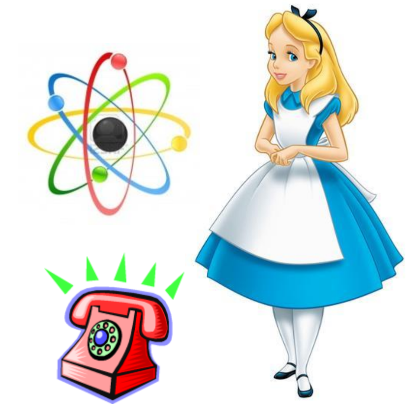
**Asymptotic additivity:** $E(\rho) = \lim_{n\to\infty} \frac{E(\rho^{\otimes n})}{n}$




**Asymptotic additivity is weaker than additivity**

# Operational measures

# Operational Measures



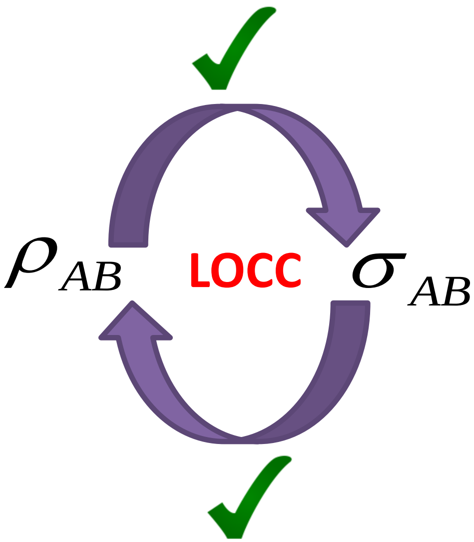
**Any arbitrary state can be made from maximally entangled states by using LOCC operations**

$$\text{If } \exists\, LOCC\ \xi:\ \sigma_{AB} = \xi(\rho_{AB}) \Rightarrow E(\sigma_{AB}) \leq E(\rho_{AB})$$

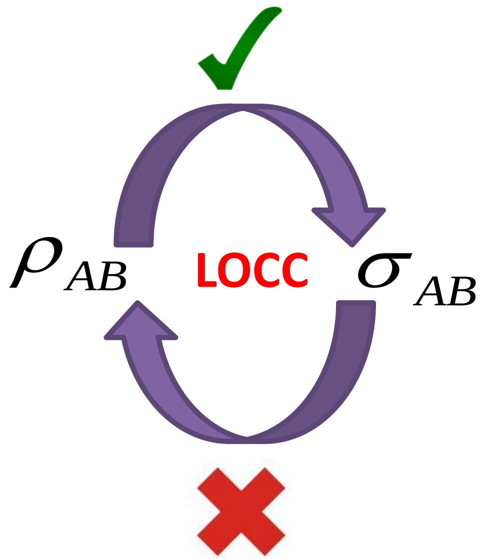
↑

**Maximally entangled**

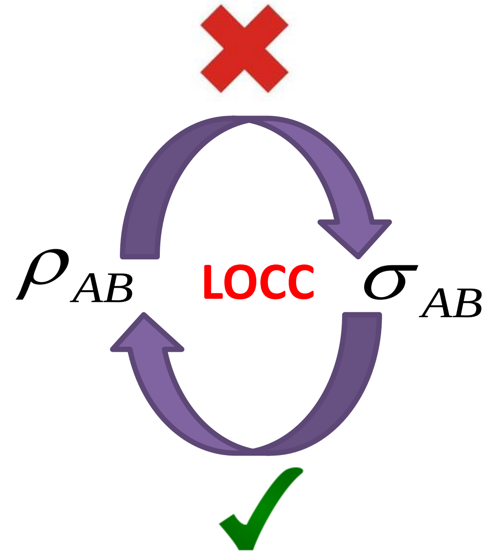
# Ordering the Quantum States (Single Copy)




$$E(\rho_{AB}) = E(\sigma_{AB})$$

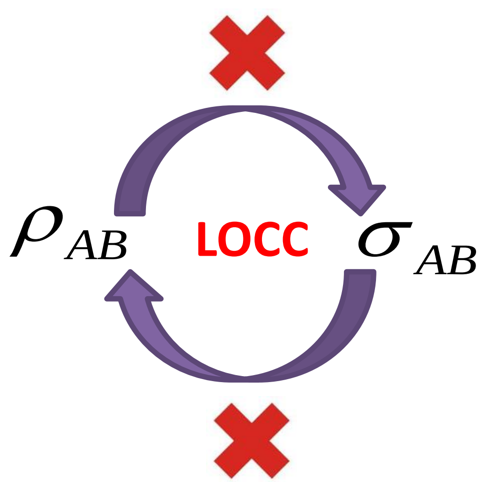



$$E(\rho_{AB}) > E(\sigma_{AB})$$



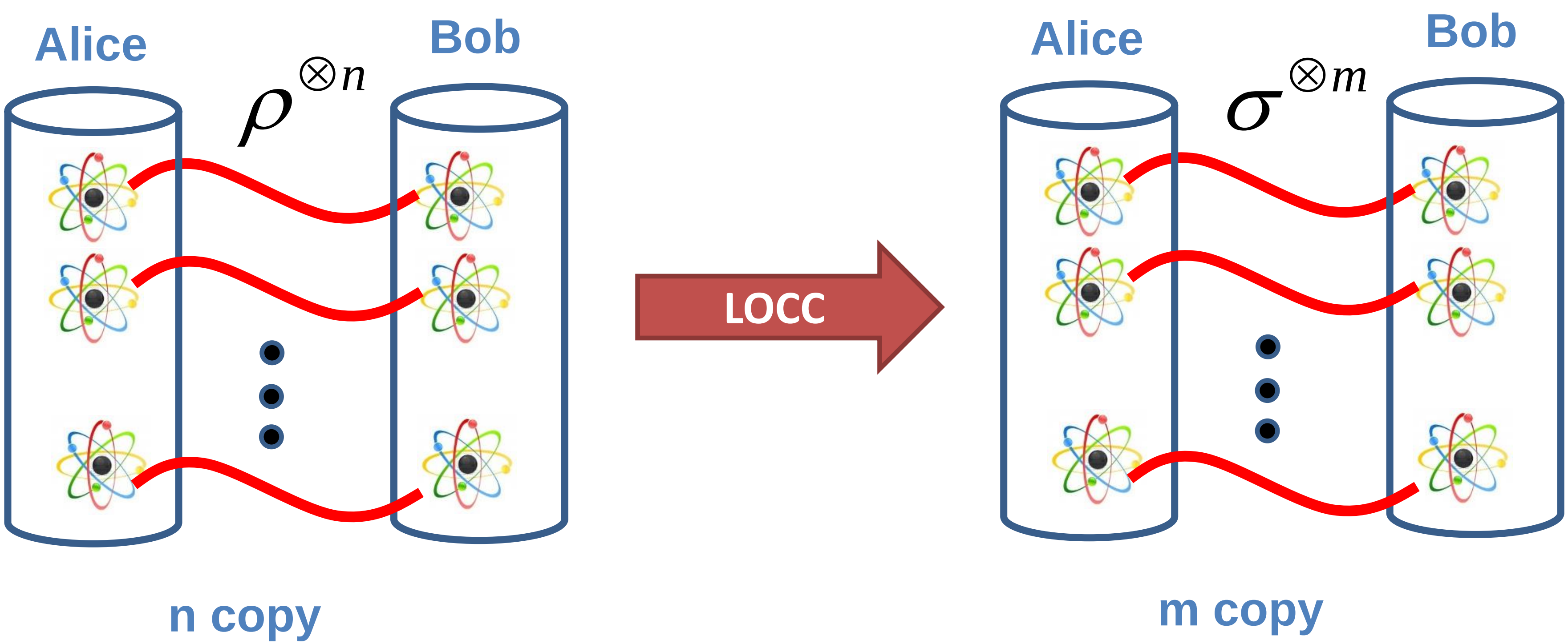

$$E(\rho_{AB}) < E(\sigma_{AB})$$

**Are LOCC maps enough for giving order to quantum states? NO !!**




$$E(\rho_{AB}) \; ? \; E(\sigma_{AB})$$

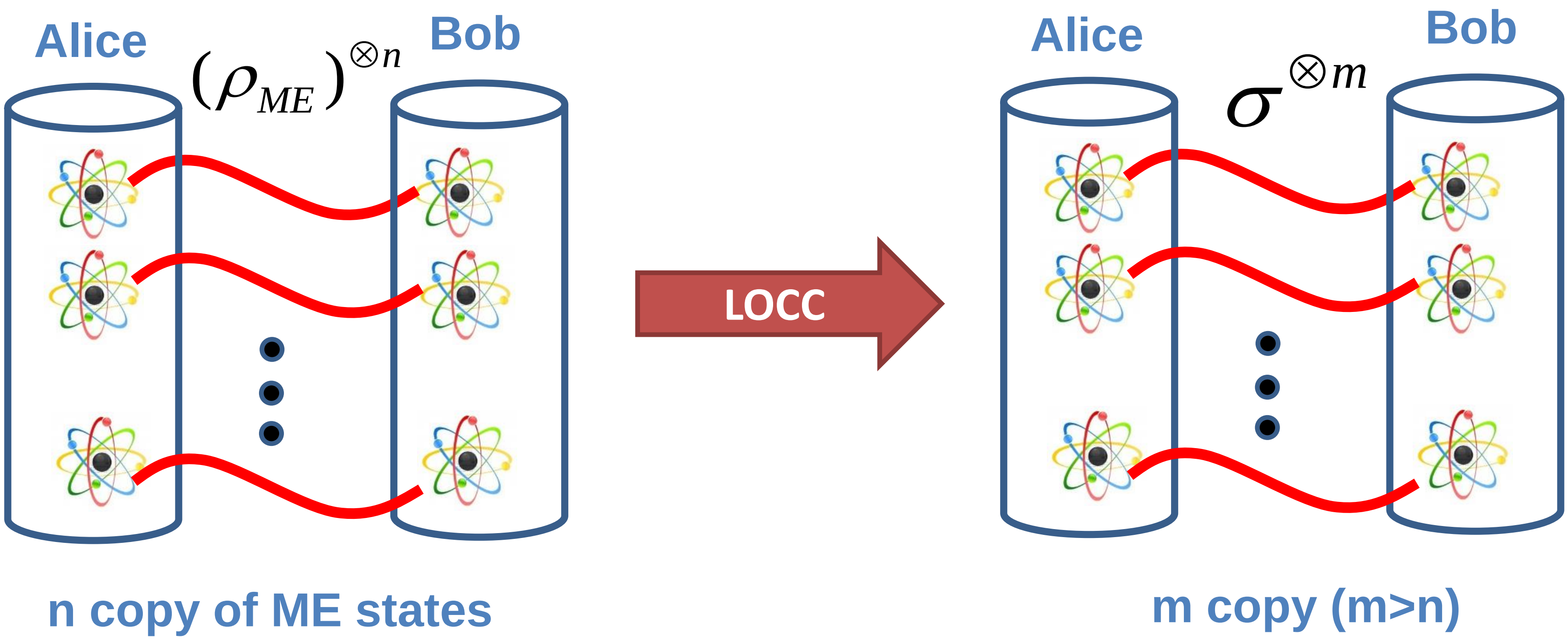
# Asymptotic Approach




Having many copies allows for more complex operations

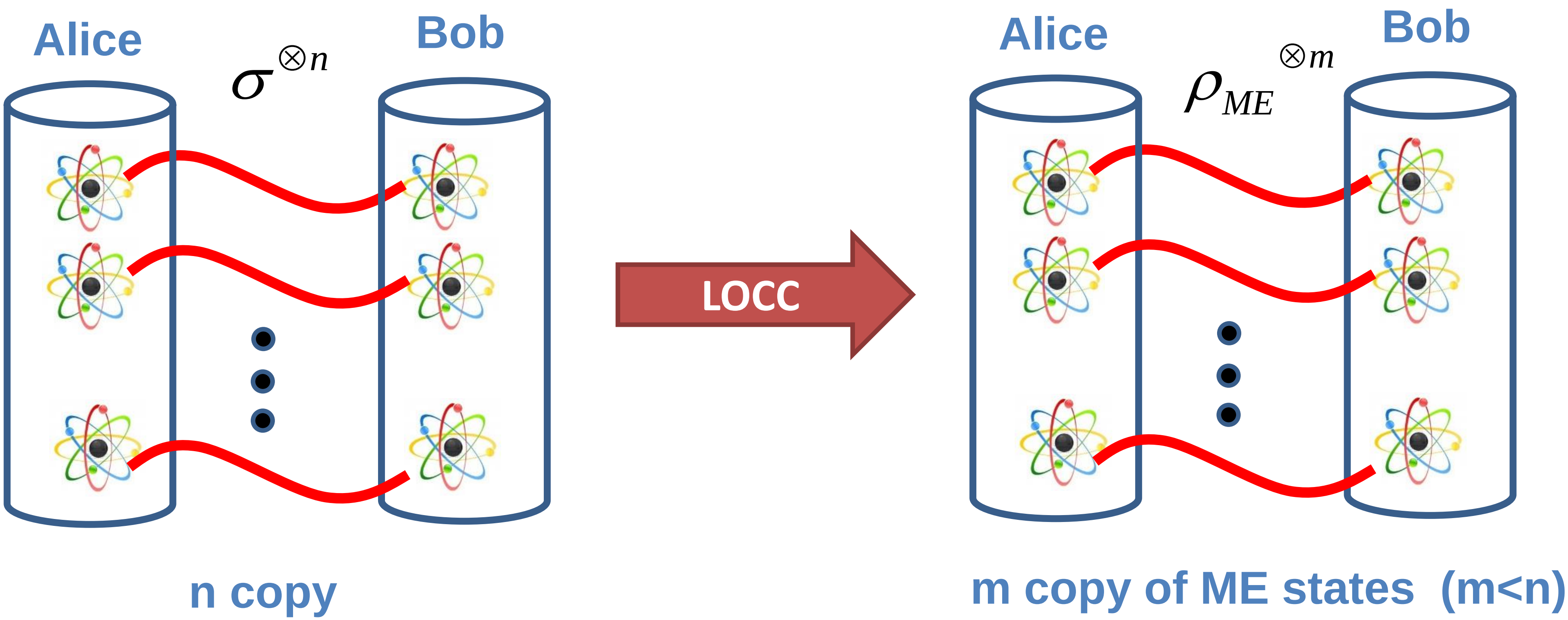
$$\frac{E(\rho_{AB})}{E(\sigma_{AB})} = \underset{LOCC}{Sup}\left\{\lim_{n\to\infty} \frac{m}{n}\right\}$$

# Entanglement Cost



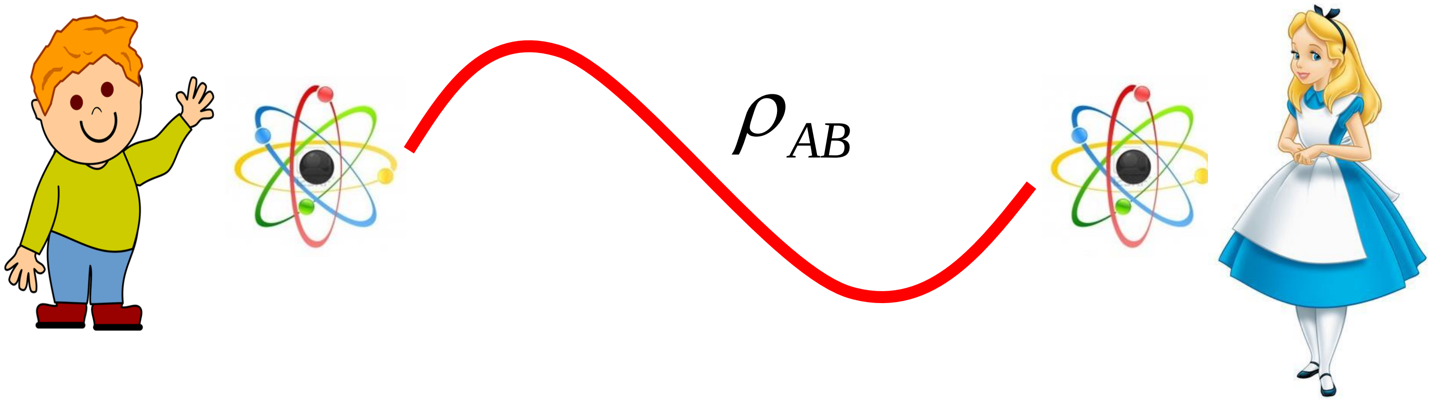

$$E_C(\sigma) = \underset{LOCC}{Sup}\left\{\lim_{n\to\infty}\frac{n}{m}\right\}$$

# Entanglement Distillation (Concentration)




$$E_D(\sigma) = \underset{LOCC}{Sup}\left\{\lim_{n\to\infty}\frac{m}{n}\right\}$$

# Non-Distillable Entanglement




**Non-distillable or bound entanglement:**

$$\begin{cases} \rho \neq \sum_i p_i \rho_i^A \otimes \rho_i^B \\ E_C(\rho) > 0 \\ E_D(\rho) = 0 \end{cases}$$

**Bound entanglement (or PPT states) cannot be used for teleportation or dense coding**

# Entanglement Cost vs. Entanglement Distillation

For pure states:

$$\begin{cases} Tr(\rho_{AB}{}^{2}) = 1 \\ E_C(\rho_{AB}) = E_D(\rho_{AB}) = S(\rho_A) = S(\rho_B) \end{cases}$$

C. H. Bennett, H. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A 53, 2046 (1996).

That is why von Neumann entropy of the subsystem is considered as the unique measure of entanglement for pure states

# Entanglement Cost vs. Entanglement Distillation

For an **arbitrary entanglement measure L** which is asymptotically additive we have:

$$E_C(\rho_{AB}) \geq \lim_{n\to\infty} \frac{L(\rho_{AB}{}^{\otimes n})}{n} \geq E_D(\rho_{AB})$$

M.J. Donald, M. Horodecki, and O. Rudolph, J. Math. Phys. 43, 4252 (2002).

Entanglement cost and entanglement distillations are extremal measures.

# Asymptotic measures

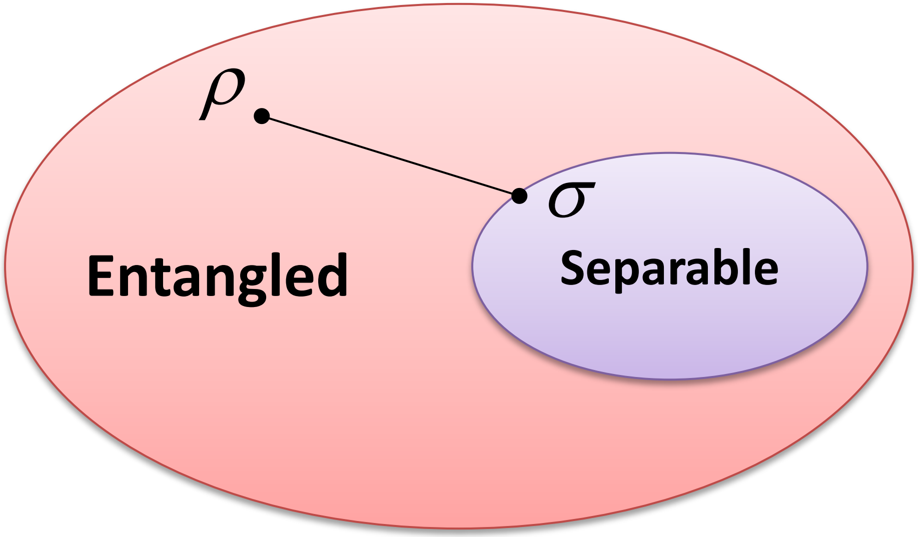
# Axiomatic Measures

Axiomatic measures:
- Distance measures
- Entanglement of formation
- Negativity

# Distance Based Measures

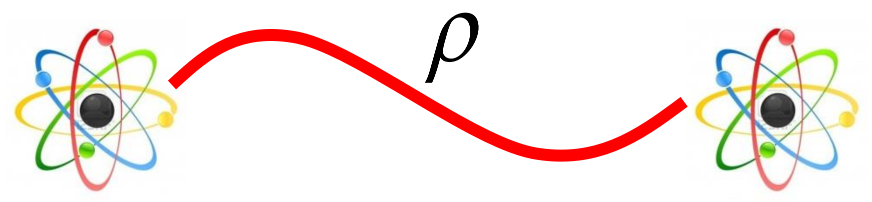
$$E(\rho) = \inf_{\sigma \in SEP} D(\rho, \sigma)$$

The set of all states




D is a distance function and can be considered as relative entropy or trace norm distance.

# Entanglement of Formation




$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Pure state

**Is it possible to quantify entanglement as:** $E(\rho) = \sum_i p_i E\left(|\psi_i\rangle\langle\psi_i|\right)$

**No! Because decomposition of a density matrix is not unique**

**Example:** $|\psi_1\rangle_{AB} = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle_{AB} = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}$

$$\rho_{AB} = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$$

$$E(|\psi_1\rangle) = 1 \quad E(|\psi_2\rangle) = 1 \quad E(|00\rangle) = 0 \quad E(|11\rangle) = 0$$

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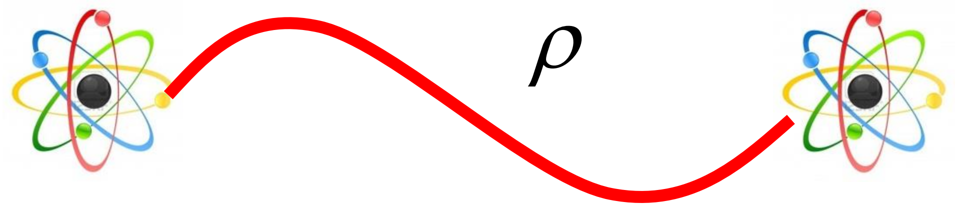



**There are infinite decompositions:**

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$E_F(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i E(|\psi_i\rangle\langle\psi_i|) : \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \right\}$$

# Concurrence

**Entanglement of formation has been computed for two qubits:**

$$E(\rho) = -\frac{1+\sqrt{1-C^2}}{2}\log\left(\frac{1+\sqrt{1-C^2}}{2}\right) - \frac{1-\sqrt{1-C^2}}{2}\log\left(\frac{1-\sqrt{1-C^2}}{2}\right)$$

**C is called concurrence and can be computed as:**

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \Rightarrow R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}} \Rightarrow$$

$$\{\lambda_i : \text{for } i > j: \ \lambda_j > \lambda_i\} \Rightarrow C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

**C is a monotonic function of C and thus usually concurrence is also used as an entanglement measure**

**W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998)**

# Asymptotic Definition

$$E_F^{\infty}(\rho) = \lim_{n\to\infty} \frac{E_F(\rho^{\otimes n})}{n} = E_C(\rho)$$

P. Hayden, M. Horodecki, and B.M. Terhal, J. Phys. A 34, 6891 (2001).

Entanglement of formation is not additive and thus:

$$E_F(\rho) \neq E_F^{\infty}(\rho)$$

M. B. Hastings, Nature Physics 5, 255 (2009)

# Transpose of a Density Matrix

**Density matrix:** $\rho = \sum_{i,j} \rho_{i.j} |i\rangle\langle j|$ $\Rightarrow$

$$\begin{cases} \rho = \rho^{+} \\ Tr(\rho) = 1 \\ \rho \geq 0 \end{cases}$$

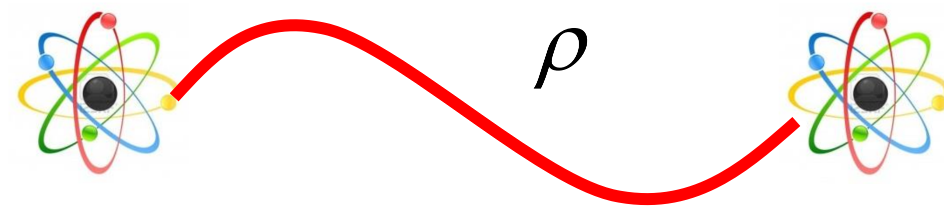
**The transpose of the density matrix is also a valid density matrix:**

$\rho^{t} = \sum_{i,j} \rho_{i.j} | j\rangle\langle i|$ $\Rightarrow$

$$\begin{cases} \rho^{t} = (\rho^{t})^{+} & ✓ \\ Tr(\rho^{t}) = 1 & ✓ \\ \rho^{t} \geq 0 & ✓ \end{cases}$$

# Partial Transpose of a Density Matrix

**Bipartite System:**




**Density matrix:** $$\rho = \sum_{i,j} \rho_{ij,kl} |i_A\rangle\langle k_A| \otimes |j_B\rangle\langle l_B|$$

**The partial transpose of the density matrix is defined as:**

$$\rho^{T_A} = \sum_{i,j} \rho_{ij,kl} |k_A\rangle\langle i_A| \otimes |j_B\rangle\langle l_B|$$

$$\rho^{T_B} = \sum_{i,j} \rho_{ij,kl} |i_A\rangle\langle k_A| \otimes |l_B\rangle\langle j_B|$$

**Properties:**

- $\rho^{T_A} = (\rho^{T_A})^+$ ✓
- $Tr(\rho^{T_A}) = 1$ ✓
- $\rho^{T_A} \geq 0$ ✗

# Negativity

**Separable:**

$$\rho = \sum_i p_i \, \rho_i^A \otimes \rho_i^B \;\Rightarrow\; \rho^{T_A} = \sum_i p_i \, (\rho_i^A)^t \otimes \rho_i^B \;\Rightarrow\; \rho^{T_A} \geq 0 \; \checkmark$$

Valid density matrix

**Entangled:**

$$\rho^{T_A} |\lambda\rangle = \lambda |\lambda\rangle \quad (\lambda < 0)$$

**Negativity:**

$$N(\rho) = 2 \sum_{\lambda<0} |\lambda|, \qquad \rho^{T_A} |\lambda\rangle = \lambda |\lambda\rangle$$

# PPT States

$$\rho \neq \sum_i p_i \, \rho_i^A \otimes \rho_i^B, \qquad \rho^{T_A} \geq 0$$

**Non-distillable entangled states**

# Entanglement Witness

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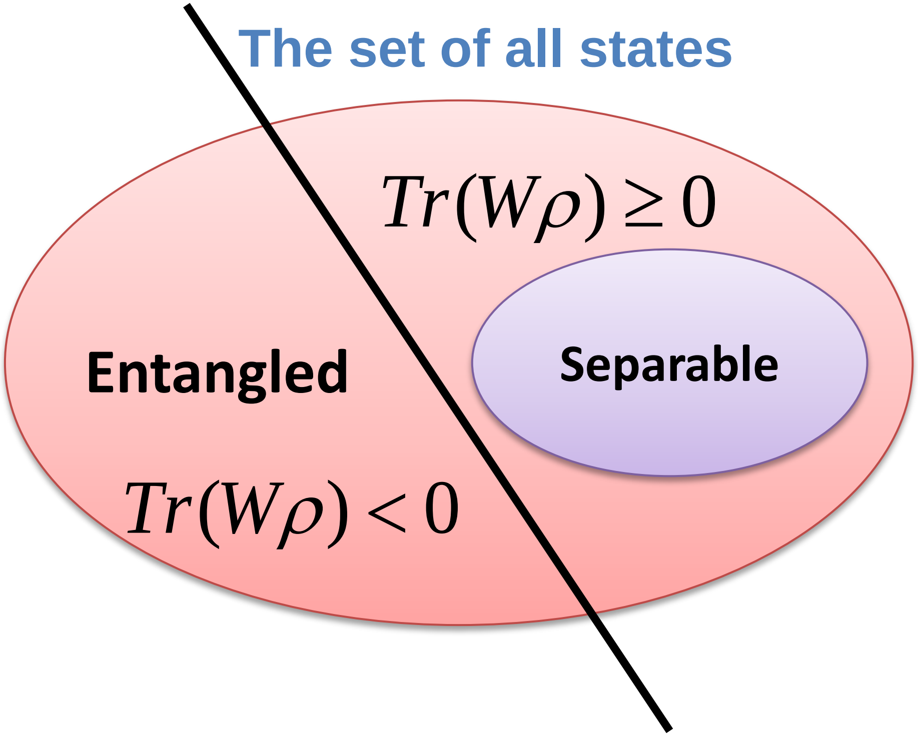


**A Hermitian operator W is entanglement witness if:**

$$\begin{cases} \forall \rho \in SEP: \quad Tr(W\rho) \geq 0 \\ \& \\ \exists \rho \notin SEP: \quad Tr(W\rho) < 0 \end{cases}$$

**Example: CHSH Bell inequality is a well known entanglement witness**

# Summary

**Pure states: von Neumann entropy**

**Mixed states: There is not a unique measure**

- **Entanglement cost (upper bound)**
- **Entanglement distillation (lower bound)**
- **Entanglement of formation (for qubits concurrence)**
- **Distance entanglement**
- **Negativity (easily computable)**

**Except negativity the rest of measures are often notoriously difficult to compute**

# Good Reviews

1- An introduction to entanglement measures
M. B. Plenio, S. Virmani
Quantum Information and Computation Vol:7, 1-51, 2007

2- Entanglement measures
M. Horodecki
Quantum Information and Computation Vol:1, 3-26, 2001