INTEGER QUANTUM HALL EFFECT: HALL CONDUCTANCE AND CHERN NUMBERS

In our discussion of the SSH model we saw how we could connect a topological invariant (winding number = $\frac{\text{Zak phase}}{\pi}$) to electric polarization P. However, as we also noted, P is only defined mod e (since the Berry phase (= Zak!) is defined only mod 2π) and is not an observable. Only changes of P are observable!

This raises the question: Are there examples where a topological invariant directly enters an observable? The answer is YES!

This will take us to the study of the INTEGER QUANTUM HALL EFFECT (IQHE) and CHERN INSULATORS. The IQHE is the mother of topological quantum phases, experimentally discovered in the early 80's (von Klitzing 1980) and given a theoretical wrapping in terms of topology by Thouless et al. (TKNN 1982) for which Thouless was partly awarded a Nobel prize in physics 2016. (He was also awarded for his work with Kosterlitz from the 70's on defect-driven classical topological phase transitions.)

As for CHERN INSULATORS, they were proposed theoretically in 1988, one model of such an insulator on graphene ("layer of graphite") by Duncan Haldane, who shared the Nobel prize with Thouless and Kosterlitz in 2016. (However, not for this work, but for his work on the physics of spin chains, showing that they can also be understood in terms of topology. (Maybe the prize committee wanted to save a second prize for Haldane because his work on the Chern insulator did prove to be instrumental...)

Starting with the IQHE, let me begin by showing you a few

{slides} just as a warm-up. Hall & Chern intro

So, we need to do some work to understand this!

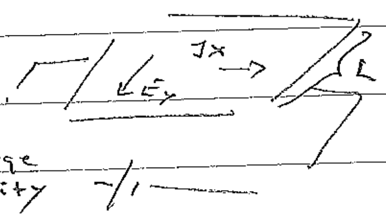
To set the stage, let me introduce some basic concepts that we need, taking off from the classical Hall effect.

Let's begin by deriving Hall's formula for the transverse (Hall) resistance $R_H = V_y / I_x = \frac{LE_y}{LJ_x} = \frac{E_y}{J_x} = \rho_{xy} \quad (= -\rho_{yx})$

from rotational invariance

$$R_H = \frac{B}{q n_{ee}} \qquad \text{(III.1)}$$

$q = -e$; n_{ee} charge density



R_H is insensitive to impurity scattering!

Drude model of dissipative transport in metals:

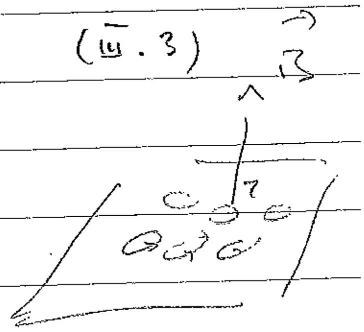
$$\dot{\vec{p}} = -e\left(\vec{E} + \frac{\vec{p}}{m} \times \vec{B}\right) - \frac{\vec{p}}{\tau} \qquad \text{(III.2)}$$

τ = scattering time (from impurity scattering)

$\vec{B} = \vec{E} = 0 \Rightarrow \vec{p}(t) = \vec{p}_0 e^{-t/\tau}$

stationary system in 2D with $\vec{B} = B\hat{z}$

$$\begin{cases} eE_x = -\frac{eB}{m} p_y - \frac{p_x}{\tau} \\ eE_y = \frac{eB}{m} p_x - \frac{p_y}{\tau} \end{cases} \qquad \text{(III.3)}$$



Introducing the cyclotron frequency

$$\omega_c = \frac{eB}{m}, \qquad \text{(III.4)}$$

and Drude's result for the longitudinal conductivity in the absence of a magnetic field

$$\sigma_0 = \frac{n_{ee} e^2 \tau}{m}, \qquad \text{(III.5)}$$

it follows from (III.3) that

$$\begin{cases} \sigma_0 E_x = -en_{ee}\frac{p_x}{m} - en_{ee}\frac{p_y}{m}\omega_c\tau \\ \sigma_0 E_y = en_{ee}\frac{p_x}{m}\omega_c\tau - en_{ee}\frac{p_y}{m} \end{cases} \qquad (III.6)$$

Recall that the electronic current density is given by

$$\vec{j} = -en_{ee}\frac{\vec{p}}{m} \qquad (III.7)$$

with

$$\vec{E} = \rho\,\vec{j} \qquad (III.8)$$

($\mu = v_0/E$)

$\mu = \frac{e\tau}{m}$ mobility

It follows that the presence of the magnetic field implies a matrix structure connecting current and electric field: $\rho = \sigma^{-1} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & \mu B \\ -\mu B & 1 \end{pmatrix} \Rightarrow$ (*

$\vec{\ }$ inverse conductivity tensor in presence of a magnetic field

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \rho\begin{pmatrix} j_x \\ j_y \end{pmatrix} \qquad = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \qquad (III.9)$$

Note that the Hall resistivity ρ_{xy} is independent of τ, i.e. insensitive to impurity scattering, cf. Eq. (III.1).

Further note that

$$\sigma = \rho^{-1} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} = \frac{1}{\rho_{xx}^2+\rho_{xy}^2}\begin{pmatrix} \rho_{xx} & -\rho_{xy} \\ \rho_{xy} & \rho_{xx} \end{pmatrix} \qquad (III.10)$$

$$(^* \quad \frac{1}{\sigma_0}\mu B = \frac{1}{n_{ee}e^2\tau}\,m\,e\,\frac{\tau}{m}B = \frac{B}{en_{ee}} = R_H)$$

HALL'S FORMULA FROM 1879

no magnetic field

If $\rho_{xy} = 0$ we get $\sigma_{xx} = \rho_{xx}^{-1}$, as expected.

However, if $\rho_{xy} \neq 0$ we obtain

magnetic field!

$$\rho_{xx} = 0 \Rightarrow \sigma_{xx} = 0 \qquad (\text{III}.11)$$

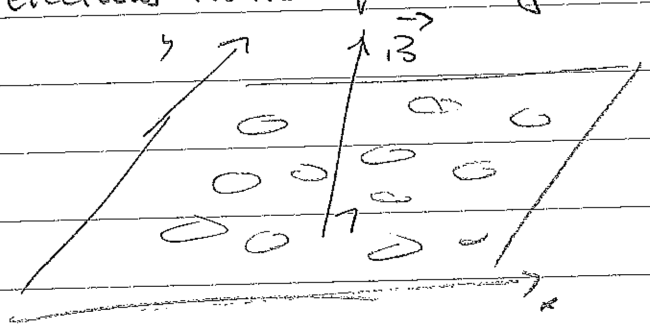
$\tau \to \infty$: no scattering
current is flowing perpendicular to the applied electric field
$\vec{E}\cdot\vec{j} = 0$
⇓
no work
⇓
no dissipation

no dissipation of energy
"like in a perfect conductor")

no current flowing in response to E_x
"like in a perfect insulator")

QUANTUM!

Let's now turn to the IQHE. First, some preliminaries about 2D electrons in a magnetic field $\vec{B} = \nabla \times \vec{A}$.



("Peierls substitution")

Gauge-invariant momentum $\vec{\Pi}$ via "minimal coupling":
(Jackson!)

$$\vec{p} \to \vec{\Pi} = \vec{p} + e\vec{A} \qquad (\text{III}.12)$$

Gauge invariance of $\vec{\Pi}$ follows from

$$\begin{cases} \vec{A}(\vec{r}) \to \vec{A}(\vec{r}) + \nabla\lambda(\vec{r}) \\ \vec{p} \to \vec{p} - e\nabla\lambda(\vec{r}) \end{cases} \qquad (\text{III}.13)$$

≈ 25 nm if B ≈ 30 T

Å

(III.12) is OK on a lattice if $a \ll \ell_B = \sqrt{\frac{\hbar}{eB}}$

lattice spacing

"magnetic length"

(physical meaning: smallest size of a cyclotron orbit allowed by the uncertainty relation, Hamiltonian for a non-interacting electron (neglecting spin due to the large Zeeman splitting from the strong magnetic field):

$$H_B = \frac{1}{2m}|\vec{\Pi}|^2 = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) \qquad (\text{III}.14a)$$

(III.5b

Choosing a Landau gauge

$$\vec{A}_L = -yB\hat{x} \qquad (\text{III}.15)$$

it follows that *)

$$[\Pi_x, \Pi_y] = -i\frac{\hbar^2}{\ell_B^2}, \qquad (\text{III}.16)$$

i.e. Π_x and Π_y are mutually conjugate $\Rightarrow$ Π_x (Π_y) generates a "boost" of Π_y (Π_x).
(Cf. "boosting" of x by p_y, being canonically conjugate.)

Introduce ladder operators (playing the role of quantized complex gauge-invariant momenta)

$$a = \frac{\ell_B}{\sqrt{2}\hbar}(\Pi_x - i\Pi_y), \quad a^+ = \frac{\ell_B}{\sqrt{2}\hbar}(\Pi_x + i\Pi_y) \qquad (\text{III}.17)$$

satisfying

$$[a, a^+] = 1 \qquad (\text{III}.18)$$

(Cf. the quantum-mechanical treatment of the 1D harmonic oscillator.)

*) $[\Pi_x, \Pi_y] = e(\underbrace{[p_x, A_y]}_{=0 \text{ in Landau gauge}} - [p_y, A_x]) = -e\left(\frac{\partial A_x}{\partial x}\cancel{[p_y, x]}^{=0} + \underbrace{\frac{\partial A_x}{\partial y}}_{-B}\underbrace{[p_y, y]}_{-i\hbar}\right) = -ie\hbar B = -i\frac{\hbar^2}{\ell_B^2}$.

use $[O_1, f(O_2)] = \frac{\partial f}{\partial O_2}\underbrace{[O_1, O_2]}_{\text{ok if c-number}}$

$H_{\vec{B}}$ is a NON-RELATIVISTIC Hamiltonian. What about its RELATIVISTIC partner?

In my last lecture (26/11) I reviewed Dirac theory, with the 2D Dirac Hamiltonian

cf (*) p. I.40 of my lecture notes

$$H = c\vec{p}\cdot\vec{\alpha} + m\beta c^2 \qquad (\text{III}.14b)$$

$$\vec{\alpha} = (\sigma_x, \sigma_y) \ , \ \beta = \sigma_z$$

Applied to (undoped) graphene (in its semimetallic phase), $m = 0$ ($m \sim$ gap) and we obtain

$$H = v\vec{p}\cdot\vec{\alpha} = v\begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} \qquad (\text{III}.14c)$$

where v is an effective parameter $\sim$ hopping amplitude $\rightarrow$ ~) $\delta = \sqrt{\kappa A}$

in the presence of a magnetic field

By doing minimal coupling $\vec{p} \rightarrow \vec{p} + e\vec{A} = \vec{\Pi}$, we finally obtain

$$H_{\vec{B}} = v\begin{pmatrix} 0 & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & 0 \end{pmatrix} \qquad (\text{III}.14d)$$

which is the starting point for studying the quantum Hall effect in graphene.

Using (III.17), (III.18), and $\omega_c = \hbar / m \ell_B^2$ one obtains

$$H_B = \hbar\omega_c \left(a^\dagger a + \tfrac{1}{2} \right) \qquad (\text{III}.19)$$

As for the 1D harmonic oscillator, the eigenvalues and eigenstates of H_B are those of the "number operator" $a^\dagger a$, with $a^\dagger a|n\rangle = n|n\rangle$, and it follows that

LANDAU LEVELS

$$\mathcal{E}_n = \hbar\omega_c \left(n + \tfrac{1}{2} \right) \qquad (\text{III}.20)$$

These levels are massively degenerate! See below!

(Note: the Hilbert space can be constructed by using $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $a|n\rangle = \sqrt{n}\,|n-1\rangle$, $n \geq 1$, $a|0\rangle = 0$)

We can look at all this from a slightly different angle, allowing us in particular to determine the degeneracy of a Landau level (= # of available states in a Landau level)

First note that the Landau gauge (III.15) breaks translational invariance in the y-direction. But translational invariance along x is OK → $p_x = \hbar k$ is a "good quantum number". We can then make the Ansatz for energy eigenfunctions of H_B in (III.14):

$$\Psi_{n,k}(x,y) \sim e^{ikx}\, \Phi_{n,k}(y) \qquad (\text{III}.21)$$

Landau level index

Acting with $H_B = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) = \frac{1}{2m}((p_x - eyB)^2 + p_y^2)$, using that $\underset{\text{operator!}}{p_x}\, e^{ikx} = \hbar k e^{ikx}$, it follows that

$$H_B \Psi_{n,k}(x,y) = \frac{1}{2m}((p_x - eyB)^2 + p_y^2)\Psi_{n,k}(x,y)$$

$$= \frac{1}{2m}((\hbar k - eyB)^2 + p_y^2)\Psi_{n,k}(x,y)$$

$$\equiv H_k \Psi_{nk}(x,y) \qquad (\text{III}.22)$$

$\omega_B = eB/m$

$$\text{where } H_k = \frac{1}{2m}p_y^2 + \frac{m\omega_B^2}{2}(y - kl_B^2)^2 \qquad (\text{III}.23)$$

$l_B = \sqrt{\frac{\hbar}{eB}}$

is the Hamiltonian of a 1D harmonic oscillator with center displaced by kl_B^2 from the origin

$\Downarrow$

$$\underset{\omega_c}{E_n} = \hbar\omega_B(n + 1/2) \quad \text{(as we've already found out!)}$$

$$\Psi_{n,k}(x,y) \sim e^{ikx}\underbrace{H_n(y - kl_B^2)}_{\text{with Hermite polynomial}} e^{-(y-kl_B^2)^2/2l_B^2} \qquad (\text{III}.24)$$

The energy eigenvalues depend only on n. Hence, the degeneracy of a Landau level is given by the # of allowed k-values.

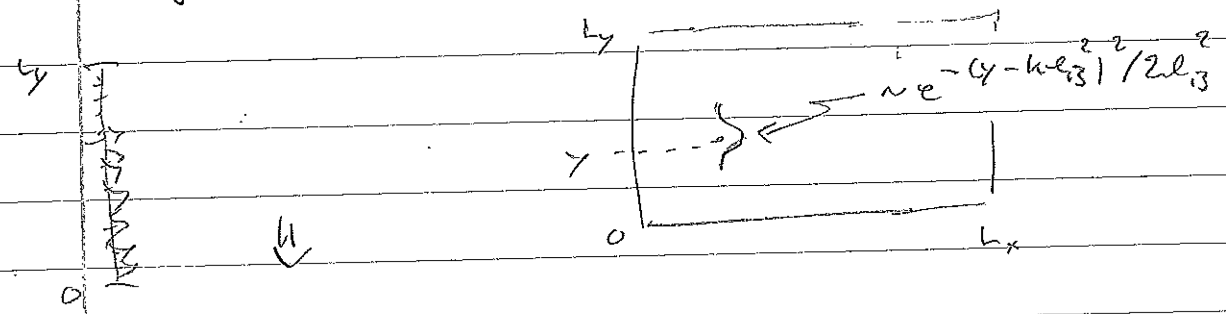
Comment: $\Psi_{n,k}(x,y)$ look like strips, no resemblance to cyclotron orbits! But these can be obtained (if wanted) by exploiting the large degeneracy and form proper linear combinations of the strips

To determine the degeneracy, consider a rectangle of lengths L_x and L_y in the (x,y)-plane. How many states fit inside this rectangle?

"Particle in a box" argument for the x-direction:

$$\Delta k = \frac{2\pi}{L_x} \qquad (\text{III}.25)$$

A wave function with a given k is exponentially localized around $y = k\ell_B^2$. Given the rectangle with length L_y along the y-direction we thus expect the k-values to range between $0 \lesssim k \lesssim L_y/\ell_B^2$.



\# of states in a Landau level:

$$N = \sum_{j=1}^{N} 1 = \frac{L_x}{2\pi}\sum_k \Delta_k \rightarrow \frac{L_x}{2\pi}\int_0^{L_y/\ell_B^2} dk = \frac{L_x L_y}{2\pi \ell_B^2}$$

$\ell_B = \sqrt{\frac{\hbar}{eB}}$

$$= \frac{eB L_x L_y}{2\pi\hbar}$$

$A = L_x L_y$

$$= \frac{BA}{\Phi_0} \qquad (\text{III}.26a)$$

$\Rightarrow n = \frac{B}{\Phi_0}$ $\quad n = \text{DOS}$ (Density of States $= \#$ of states/unit area)

where $\Phi_0 = \frac{2\pi\hbar}{e}$ is called the "flux quantum"

(= magnetic flux contained within an area $2\pi\ell_B^2$ ~ magnetic flux within a cyclotron orbit)

$\Rightarrow \Phi_0 = B \cdot 2\pi\ell_B^2$

Let's assume that we fill the lowest Landau level completely:

electron density $\rightarrow$ $$n_{ee} = \overset{\text{DOS}}{n} \qquad (\text{III}.26\,b)$$

i.e. one electron / available state. It follows from (III.26 a) and (III.26 b) that

$$B = n_{ee}\,\Phi_0 \qquad (\text{III}.26\,c)$$

Now suppose we increase B so that $n > n_{ee}$. Then the lowest Landau level is only partially filled. When applying an electric field, there are now available empty states for the electrons to scatter into. We then expect that $R_H = \rho_{xy}$ will change! Yet it stays constant!

To understand why, we need to bring in topology!

With this as a background, let us now turn to the task of computing the Hall resistivity of the IQHE.
I will do a light version of the calculation performed by Thouless et al. in 1982, working in a continuum formulation rather than on a ("magnetic") lattice.
The aim is to see how topology enters and explains the remarkable quantization of the Hall resistivity!

For this purpose, we'll need the Kubo formula for linear response.

Slides Kubo