Lecture 26/11 (I.38

SYMMETRY-PROTECTED BOUNDARY STATES.

At my last lecture I showed you the "periodic table" of symmetry-protected topological phases ("TEN-FOLD WAY"/ "ALTLAND-ZIRNBAUER CLASSIFICATION") which specifies which topological phases are possible given a perturbation that respects / does not respect one or several of the three symmetries CHIRAL, TIME-REVERSAL, and PARTICLE-HOLE symmetry. Time-reversal T and particle-hole C transformations come in two classes, $T^2 = \pm 1$ and $C^2 = \pm 1$, and this also impacts what phases are possible.

I also briefly discussed that the HALLMARK of a topological phase is the presence of ROBUST BOUNDARY STATES and that this robustness is perceived as the key to use topological phases for future quantum technologies. So, how do symmetries protect the boundary states? Let's go back to the SSH model to develop some intuition!

SLIDES

For a deeper understanding, it is useful to go to a continuum limit of the SSH model, in this way mapping it onto the Dirac Hamiltonian in 1D. This will not only allow us to establish the localized boundary states analytically, but also address the question how much charge these boundary states actually hold (cf. (I.58) and the 4-unit cell pictures on the slides!)

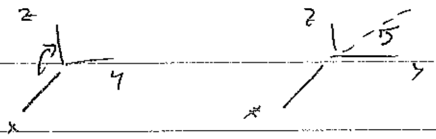
SSH $\longrightarrow$ DIRAC

Start writing the Bloch Hamiltonian of the SSH model as

ka with a=1

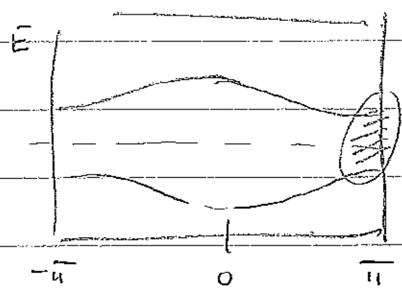
$$H(k) = \Big(\underbrace{(t+\delta t)}_{v} + \underbrace{(t-\delta t)}_{w}\cos(k)\,\sigma_x + \underbrace{(t-\delta t)}_{w}\sin(k)\,\sigma_y\Big) \qquad (I.63)$$

Do the successive transformations $\sigma_x \to \sigma_z$, $\sigma_y \to \sigma_x$



and consider $k \approx \pi - q$, $q \ll 1$

Also choose $|\delta t| \ll t = 1$



$\cos(\pi - q) \approx -1$

$\sin(\pi - q) \approx q$

$$\Downarrow$$

$$H(k) \longrightarrow \underbrace{q}_{-i\partial_x}\sigma_x + \underbrace{2\delta t}_{m}\sigma_z = -i\partial_x\sigma_x + m\sigma_z$$

we have put $c = v = 1$ and $\hbar = 1$

"1D DIRAC HAMILTONIAN"
$$= \begin{pmatrix} m & -i\partial_x \\ -i\partial_x & -m \end{pmatrix} \qquad (I.64)$$

Let's back up by deriving the Dirac Hamiltonian from the Dirac Lagrangian by a Legendre transformation (for those of you more familiar with QFT)

we will specialize to 1+1 shortly!

Start with the well-known Dirac equation in 3+1 D

4-spinor *

$$(i\partial\!\!\!/ - m)\psi = 0, \quad \partial\!\!\!/ = \gamma^\mu\partial_\mu$$

* I write ψ for the wave-function and $\hat{\Psi}$ for the operator. See below.

from Euler-Lagrange, using the Dirac Lagrangian

DIRAC MATRICES
$\gamma^0 = \beta = \sigma_z \otimes \mathbb{1}$
$\gamma^i = \beta\alpha^i = (\sigma_z\otimes\sigma_0)(\sigma_x\otimes\sigma_i)$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \quad \bar{\psi} = \psi^\dagger\gamma^0$$

Now do a Legendre transformation

$$\mathcal{H} = \Pi_\mu \partial_0 \psi^\mu - \mathcal{L}$$

Use that $\Pi_\mu = \frac{\delta \mathcal{L}}{\delta(\partial_0 \psi_\mu)} = i\psi_\mu^+$ to obtain

$$\mathcal{H} = i\psi_\mu^+ \partial_0 \psi^\mu - \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

some algebra using the expressions for the γ-matrices on the previous page

$$= -i\psi^+ \nabla\cdot\vec{\alpha}\,\psi + \beta m \psi^+\psi$$

FIRST-QUANTIZED HAMILTONIAN

$$(**) \Rightarrow \quad H = (-i\nabla\cdot\vec{\alpha}) + m\beta$$

inserting c to get dimensions right

$$= c\vec{p}\cdot\vec{\alpha} + m\beta c^2 \qquad (*)$$

In 1D $\quad \vec{\alpha} = \alpha_x,\ \beta = \sigma_z$ with ψ a 2-spinor $(**)$

2D $\quad \vec{\alpha} = (\alpha_x, \alpha_y) = (\sigma_x, \sigma_y)$ with ψ a 2-spinor

Putting $(**)$ into $(*)$ we obtain (I.64) with $c=1$.

Jackiw & Rebbi (Phys. Rev. D13, 3398 (1976)) studied the 1D Dirac Hamiltonian (I.64) with a spatially varying mass

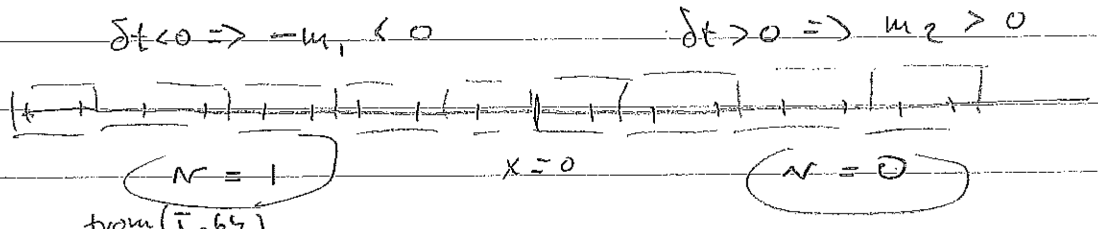
$$H = -i\partial_x \sigma_x + m(x)\sigma_z \qquad (\text{I}.65)$$

$$m(x) = \begin{cases} -m_1 & x < 0 \\ m_2 & x \geq 0 \end{cases} \qquad m_1, m_2 \geq 0$$

Standard notation 3+1D

$$(**)\quad \Psi_\alpha(x) = \text{CONST.} \int \frac{d^3p}{\underbrace{(\vec{p}^2 + m^2)^{1/2}}_{E_p = p_0}} \sum_s \Big(b(p,s)\,\underbrace{u_\alpha(p,s)}_{\text{positive energy}}\,e^{-ipx} + d^+(p,s)\,\underbrace{v_\alpha(p,s)}_{\text{negative energy}}\,e^{ipx} \Big)$$

Applied to SSH, (I.65) describes a configuration with two SSH chains glued together at $x=0$, the one to the left (right) in a topologically nontrivial (trivial) phase:



So, the eigenvalue problem we want to solve is

N.B. $\bar{\Psi}$ operator, Ψ wave function

$$\begin{pmatrix} m(x) & -i\partial_x \\ -i\partial_x & -m(x) \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = E \underbrace{\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}}_{\Psi_E(x)} \qquad (\text{I}.66)$$

label the wave function by the eigenvalue

where the operator $\bar{\Psi}_\alpha(x) = \sum_E \left(\Psi_{E,\alpha}(x) a_E + \Psi_{-E,\alpha}(x) a^\dagger_{-E} \right)$ annihilates a spinless electron at sublattice $\alpha = A, B$ in the unit cell with coordinate x.

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} \varphi_1^\pm \\ \varphi_2^\pm \end{pmatrix} e^{\mp \lambda_\pm x}$$

for $x \gtrless 0$ with $\mathrm{Re}\,\lambda_\pm > 0$ so that that $\varphi_i(x) \to 0$ as $x \to \pm\infty$

Secular equation: $\det \begin{pmatrix} m_2 - E & i\lambda_+ \\ i\lambda_+ & -m_2 - E \end{pmatrix} = 0$

$$\Rightarrow \lambda_+ = \pm\sqrt{m_2^2 - E^2} \qquad (\text{I}.67)$$

Solutions with $m_2^2 < E$ yield extended (bulk) states that we're not interested in. Instead, take $m^2 > E^2$ and pick $\lambda_+ > 0$.

Choosing $m(x) = m_2$ in (I.66) and carrying out the derivatives, one finds

$$\varphi_1^+ = -\frac{i\lambda_+}{m_2 - E}\varphi_2^+ \qquad (\text{I}.68)$$

Similarly, for $x < 0$, we have

$$\lambda_- = \sqrt{m_1^2 - E^2} \qquad (\text{I}.69)$$

and

$$\varphi_1^- = -\frac{i\lambda_-}{m_1 + E}\varphi_2^- \qquad (\text{I}.70)$$

Continuity of the wave functions at $x = 0$ requires that

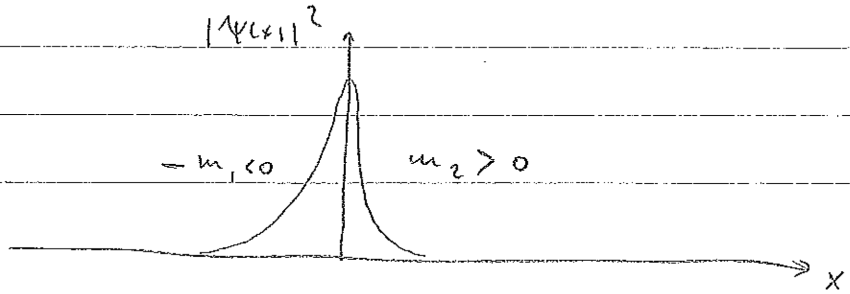
$$\begin{pmatrix} \varphi_1^+ \\ \varphi_2^+ \end{pmatrix} = \begin{pmatrix} \varphi_1^- \\ \varphi_2^- \end{pmatrix}$$

from which it follows from (I.67) - (I.70) that

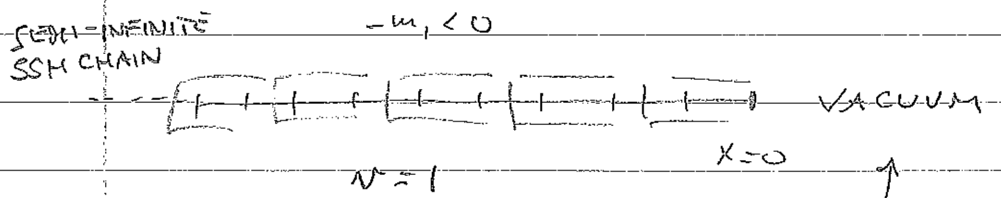
$$\frac{\sqrt{m_2^2 - E^2}}{m_2 - E} = \frac{\sqrt{m_1^2 - E^2}}{m_1 + E} \qquad (\text{I}.71)$$

$\Downarrow$ ZERO-ENERGY ($E = 0$) EXPONENTIALLY LOCALIZED SOLUTION AT THE INTERFACE

$$\psi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \text{CONST.} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-|m(x)x|} \qquad (\text{I}.72)$$

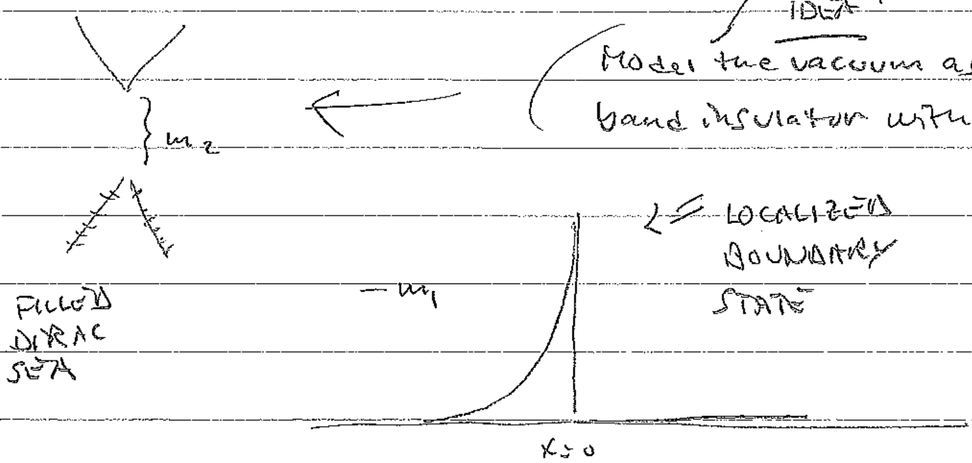


Now, what about the original problem we were interested in, with one SSH chain in the topological phase with open boundary conditions?



IDEA *

Model the vacuum as a trivial band insulator with $m_2 \to \infty$!



* To put this limit on firm ground as it comes to SSH requires some more analysis since we actually assumed that $m(x) = \delta t \ll 1$ in the parameterization of the Dirac Hamiltonian (cf. page I.39). The formal rigorous way is to model the SSH Hamiltonian by a 1D Dirac Hamiltonian with an added quadratic correction: $-\text{const.} \times \partial_x^2$

What about the robustness of the zero-energy boundary state in the Dirac picture? Let's go back to the case with two SSH chains glued together, one in a phase with $\nu = 1$ (topologically nontrivial), the other with $\nu = 0$. Assume that the major effect of a perturbation is to change the mass distribution $m(x)$.

Consider the $E=0$ solution to (I.66), now with $m(x)$ constrained only by having $m(x)$ switch sign at $x=0$, with $m(-\infty)<0$ and $m(+\infty)>0$.

It follows from (I.66): $\quad \Psi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$

$$\left(-i\partial_x \sigma_x + m(x)\sigma_z\right)\Psi(x) = 0 \qquad (I.73)$$

Multiply (I.72) by σ_x from left to obtain

$$\partial_x \Psi(x) = -m(x)\sigma_y \Psi(x) \qquad (I.74)$$

Put $\Psi(x) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \varphi(x)$:

$$\partial_x \underline{\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}} \varphi(x) = -m(x)\underline{\sigma_y \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}} \varphi(x)$$

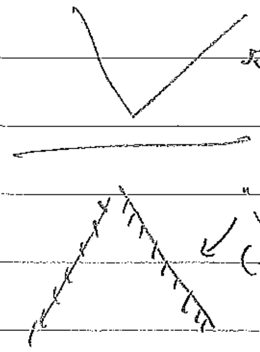
$$\Rightarrow \sigma_y \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \varphi_1^{\pm} \\ \varphi_2^{\pm} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \qquad (I.75)$$

It follows from (I.74), (I.75) that there are two solutions

$$\Psi_{\pm}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \exp\left(\int^x \pm m(x')\,dx'\right) \qquad (I.76)$$

where the signs $\pm$ are determined by the signs of $m(\pm\infty)$.
The important conclusion is that there always EXISTS A ZERO-ENERGY SOLUTION even after a deformation (perturbation!) of the mass distribution. In other words: the zero-energy solution is robust against the perturbation!

CHARGE FRACTIONALIZATION



← ~~"conduction band"~~

$E=0$

"valence band" ↙ (~~filled Dirac sea~~)

$$\begin{cases} b_E^+ & \text{particle creation operator} \\ b_E & \text{annihilation} \end{cases}$$

$$\begin{cases} d_E^+ & \text{antiparticle creation operator} \ (\text{"hole"}) \\ d_E & \text{annihilation} \end{cases}$$

$$\Psi = \sum_{E\neq 0} \left(\psi_E b_E + \psi_{-E} d_E^+ \right) + \psi_0 a$$

$$\Psi^+ = \sum_{E\neq 0} \left(\psi_E^* b_E^+ + \psi_{-E}^* d_E \right) + \psi_0^* a^+$$

↖ annihilation, ~~creation~~ ← operator for zero mode

Note that the $\pm E$ terms are "partners": We can remove/add modes with energy in two ways.

filled zero-energy state, empty zero-energy state

$$a|+\rangle = |-\rangle, \quad a^+|-\rangle = |+\rangle, \quad a|-\rangle = a^+|+\rangle = 0$$

Introduce the charge operator $Q = \int dx\, \Psi^+(x)\Psi(x)$.
Subtracting the infinite$^{(!)}$ contribution from the filled Dirac sea, we obtain the regularized expression for Q:

$$:Q: = Q - Q_{vac} \longrightarrow \quad :Q: = \frac{1}{2}\int dx \left(\Psi^+(x)\Psi(x) - \Psi(x)\Psi^+(x) \right) \qquad (I.77)$$

Using the orthonormality of the wave functions,

$$\begin{cases} \int dx\, dx'\, \psi_E^*(x)\, \psi_{-E}(x') = 0 \\ \int dx\, dx'\, \psi_E^*(x)\, \psi_E(x') = 1 \end{cases} \qquad (I.78)$$

it follows that ⟶

$$Q = \sum_{E \neq 0} \left(b_E^\dagger b_E - d_E^\dagger d_E \right) + a^\dagger a - \frac{1}{2}$$

half hole

"empty" multiply by $-e$

$$Q|-\rangle = \left(a^\dagger a - \tfrac{1}{2}\right)|-\rangle = -\frac{1}{2}|-\rangle \longrightarrow \frac{1}{2} e |-\rangle$$

$$Q|+\rangle = \left(a^\dagger a - \tfrac{1}{2}\right)|+\rangle = \frac{1}{2}|+\rangle \longrightarrow -\frac{1}{2} e |+\rangle$$

"filled" half electron