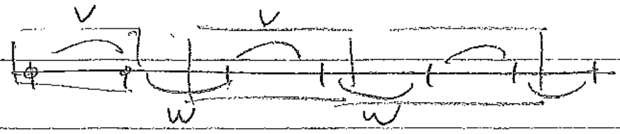
Lecture 21/11

(I.32a

SSH MODEL (CONT'D): INTRODUCTION TO SYMMETRY PROTECTED TOPOLOGICAL PHASES



$$H(k) = \vec{d}(k)\cdot\vec{\sigma} = \begin{pmatrix} 0 & d_x(k)-id_y(k) \\ d_x(k)+id_y(k) & 0 \end{pmatrix}$$

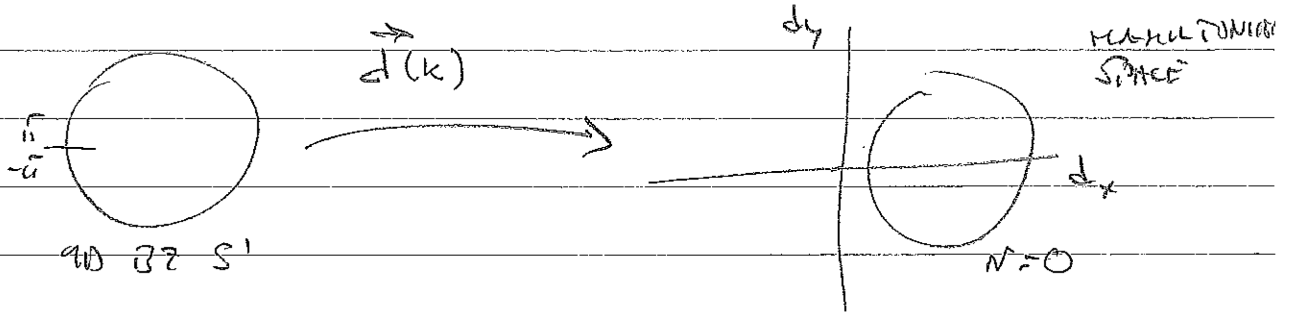
single-particle Bloch Hamiltonian

$$\begin{cases} E_\pm(k) = \pm|\vec{d}(k)| \\ |u_\pm(k)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm e^{-i\varphi(k)} \\ 1 \end{pmatrix} \end{cases} \qquad \varphi(k) = \arctan\left(\frac{d_y(k)}{d_x(k)}\right)$$

Berry phase in the lower band $\Phi_- = \begin{cases} 0 & \text{if } v>w \\ -\pi & \text{if } v<w \end{cases}$

DEFINE $\nu = -\frac{\Phi}{\pi} = \begin{cases} 0 & \text{if } v>w \\ 1 & \text{if } v<w \end{cases}$ what does it signify

WINDING NUMBER TOPOLOGICAL INVARIANT CHARACTERIZING THE MAPPING FROM $S^1 \to S^1$



ν MEASURES HOW MANY TIMES THE IMAGE OF THE 1D BZ WRAPS AROUND THE ORIGIN IN HAMILTONIAN SPACE.

Can we experimentally distinguish a topological insulator from a trivial insulator?

We can introduce an observable that enables us to distinguish between the two phases with $\nu = 0$ and $\nu = 1$, respectively: THE CHANGE OF ELECTRIC POLARIZATION ΔP, with

$$P = -e\frac{\Phi}{2\pi} \bmod e = \begin{cases} 0 \bmod e & \nu = 0 \\ e/2 \bmod e & \nu = 1 \end{cases} \qquad (\text{I.58})$$

in the TOPOLOGICAL PHASE

SLIDE!

How to observe ΔP? Cut open the 1D lattice! (So far we have assumed periodic boundary conditions.)

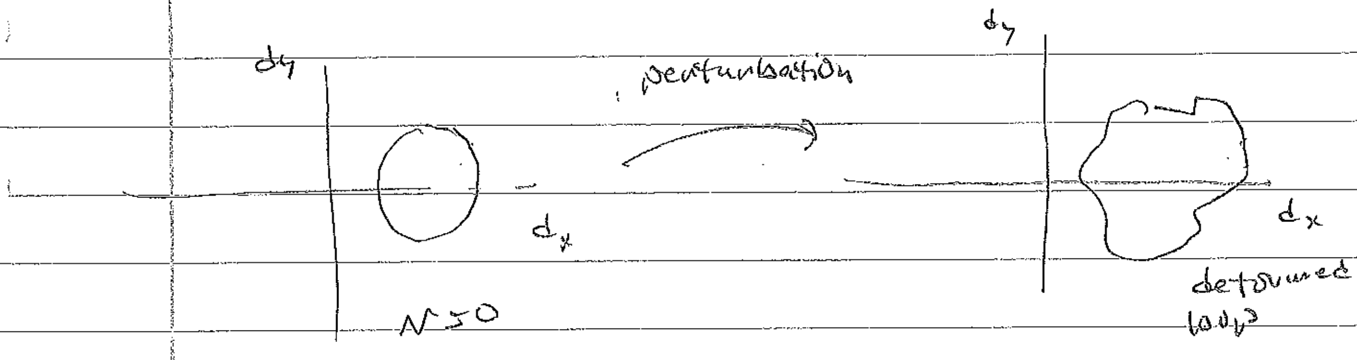
PICTURE: SLIDE!

The picture illustrating the electric polarization suggests the existence of a ZERO-ENERGY BOUNDARY STATE AT THE EDGE OF THE 1D LATTICE => TWO-FOLD DEGENERATE (since one such state at each edge) ZERO-ENERGY ENERGY LEVEL. As we shall see, such boundary states are robust against perturbations provided that these perturbations respect certain symmetries.
ROBUST BOUNDARY STATES IS THE HALLMARK OF SYMMETRY-PROTECTED TOPOLOGICAL PHASES! ALMOST ALL PROPOSALS FOR APPLICATIONS OF TOPOLOGICAL SYSTEMS (= systems supporting topological phases) FOR FUTURE QUANTUM TECHNOLOGIES TAKE OFF FROM THE ROBUST BOUNDARY STATES!

I told you before that Berry phases are not expected to be quantized

But before we proceed...

At this point you may be somewhat uncomfortable with the fact that the Berry phase ("Zak phase") is not only quantized but also defines a topological invariant, a winding number! Moreover, any* perturbation such that $\vec{d} \longrightarrow \vec{d}_{pert}$ ($\Rightarrow H(k) \longrightarrow H_{pert}(k) = H(k) + \Delta H(k) = \vec{d}_{pert} \cdot \vec{\sigma}$) will only cause a deformation of the unperturbed circles in Hamiltonian space



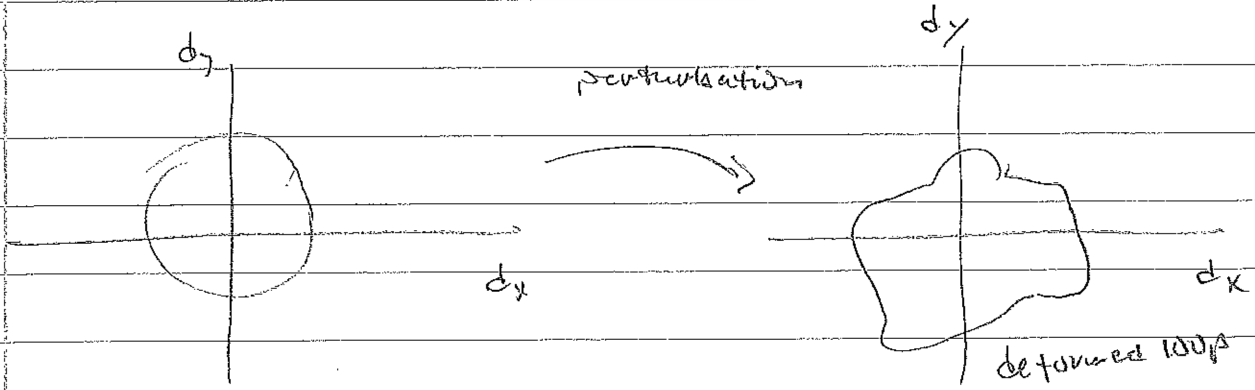
$\nu = 0$

$\nu = 0$ still!

The topological invariant is invariant under a smooth perturbation ("continuous deformation")* provided that the perturbation doesn't close the gap (in case ν is not well defined **)

**) comment on vgp?

Similarly



$\nu = 1$

$\nu = 1$ still!

This may all seem rather strange! How is it possible?

NO DIAGONAL COMPONENT OF $H_{pert}(k)$

ANSWER: In our discussion / pictures on the previous page we tacitly assumed that $\{d_{(pert)z} = 0\}$. If we allow a perturbation which tips $\vec{d}$ out of the (d_x, d_y)-plane then Φ_- starts to vary continuously. Only if $d_{(pert)z} = 0$ do we get this remarkable result!

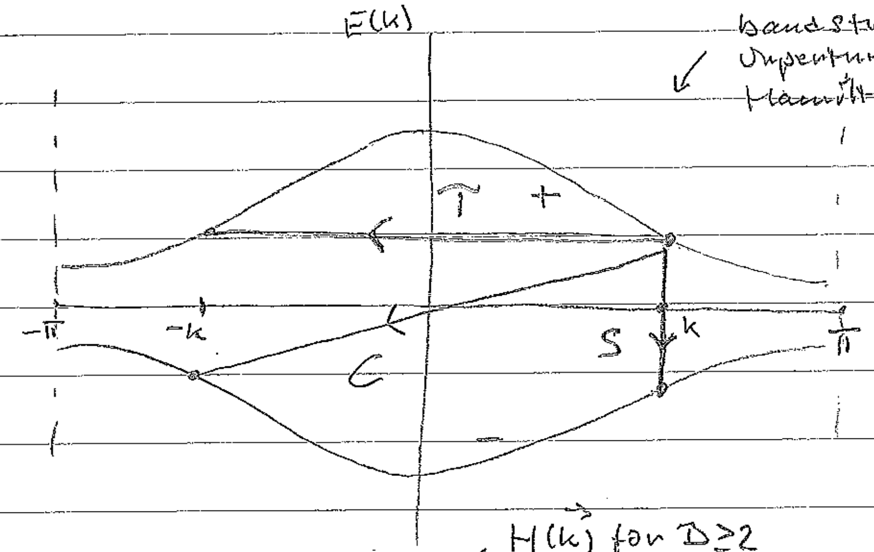
HW

But where does this constraint come from?

ANSWER: From "CHIRAL SYMMETRY" ("SUBLATTICE SYMMETRY")

Back to the BZ (periodic boundary conditions)

band structure of unperturbed SSH Hamiltonian



$H(\vec{k})$ for $D \geq 2$

A Bloch Hamiltonian $H(k)$ is said to be CHIRAL INVARIANT if there exists a transformation $S|u(k)\rangle = |u'(k)\rangle$ such that $\underbrace{|u(k)\rangle}_{|u_+(k)\rangle}$ $\underbrace{|u'(k)\rangle}_{|u_-(k)\rangle}$

This is the reason for the name chiral transformation

$$S H(k) S^{-1} = -H(k) \qquad (I.59)$$

$$\Downarrow$$

eigenvalues of $H(k)$ $\quad E(k) \xrightarrow{S} -E(k)$ (see fig. above for the unperturbed SSH Hamiltonian)

$\psi' = e^{i\alpha\gamma_5}\psi$ for 1+1D Dirac theory

$\gamma_5 = \sigma_z$ (N.B. $H(k)$ is off-diagonal in a basis where S is diagonal. In the basis we are working in we can take $S = \sigma_z$ (standard choice for 2-band models

(I.59) doesn't look like a symmetry relation. However, the − sign is an artifact from looking at the single-particle Hamiltonian

$$\hat{H} = \sum_k \Psi_k^+ \underbrace{H(k)}_{} \Psi_k \qquad \Psi_k = \begin{pmatrix} a_k \\ b_k \end{pmatrix} \text{ for unperturbed SSH Hamiltonian}$$

$$\left\{ \Psi_k \xrightarrow{\text{chiral transf.}} \Psi_k^+ \quad \text{and} \quad \underbrace{H(k) \xrightarrow{\text{chiral transf.}} -H(k)}_{SH(k)S^{-1} = -H(k)} \right\}$$

$$\Downarrow$$

$$\hat{H} \xrightarrow{\text{chiral transf.}} \hat{H}$$

i.e. $\hat{H}$ is invariant under a chiral transformation.

Going back to (I.59), in the given basis in which we have written $H(k)$ (unperturbed single-particle SSH Hamiltonian) $S = \sigma_z$.

To have a full symmetry classification of symmetry-protected phases we need to consider two more symmetries: time-reversal symmetry and particle-hole symmetry.

TIME-REVERSAL TRANSFORMATION T

$$T|u_n(k)\rangle = |u_n(-k)\rangle$$ (cf. figure on previous page for $|u_n(k)\rangle = |u_+(k)\rangle$)

$$T = U_T K$$

ANTIUNITARY (T), UNITARY (U_T), COMPLEX CONJUGATION (K): $Ki = -i$

$\langle Tu|Tv\rangle = \langle u|v\rangle^* = \langle v|u\rangle$.

H(k) is TIME-REVERSAL INVARIANT if

$$T H(k) T^{-1} = H(-k) \qquad (\mathrm{I}.60)$$

$$\Downarrow$$

$$E(k) \xrightarrow{T} E(-k)$$

Note that $T H(k) T^{-1} = (U_T K) H(k) (U_T K)^{-1}$
$= (U_T K) H(k) (K U_T^{-1}) = U_T H^*(k) \underbrace{KK}_{=1} U_T^{-1}$,
yielding the equivalent form of (I.60)

$$U_T H^*(k) U_T^{-1} = H(-k) \qquad (\mathrm{I}.61)$$

Standard choice of U_T:

$$U_T = \begin{cases} \mathbb{1}_{n\times n} & \text{for a spinless } n\text{-band model} \\ \mathbb{1}_{n\times n} \otimes (-i\sigma_y) & \text{for a spinful } 2n\text{-band model} \end{cases}$$

↑ spin flip

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} \xrightarrow{\text{ADDING SPIN}} \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \\ b_{k\uparrow} \\ b_{k\downarrow} \end{pmatrix}$$

2-band 4-band

$$\Updownarrow$$

H(k) 4x4 matrix

$\hat{Q} \xrightarrow{C} -\hat{Q}$

Finally, PARTICLE-HOLE TRANSFORMATION
("CHARGE CONJUGATION")

$$C = U_C K$$

ANTI UNITARY — UNITARY — COMPLEX CONJUGATION

$H(k)$ is PARTICLE-HOLE SYMMETRIC if

$$C H(k) C^{-1} = -H(-k) \qquad (\text{I}.62)$$

$$\Downarrow$$

$$E(k) \xrightarrow{C} -E(-k)$$

S, T, C are the three symmetries that go into the so called periodic table of noninteracting fermionic symmetry-protected topological systems ("tenfold way") = classification of possible symmetry-protected topological phases.

SLIDE

Lots of comments!

- Origin of the "periodic table"?
- Meaning of +/− BLACKBOARD
- Additional symmetries
- EXAMPLES

- Periodic structure.
- $\mathbb{Z}, \mathbb{Z}_2, 2\mathbb{Z}$
- The symmetries are constraints on allowed perturbations