(II.10 a

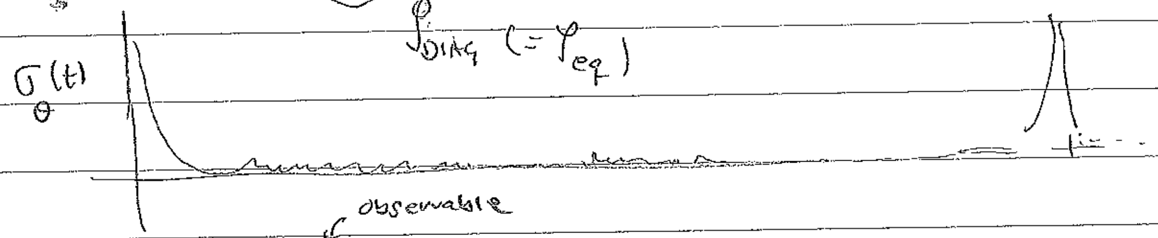
LECTURE 19 DECEMBER

First of all ...

Let's try to "wrap up" the discussion about
THERMALIZATION OF CLOSED QUANTUM SYSTEMS
from Thursday lecture ...

Recall : Out-of-equilibrium initial condition at $t=0$
(e.g. from a quantum quench) :

$$\rho(t) = \underbrace{\sum_m |c_m|^2 |m\rangle\langle m|}_{\rho_{DIAG}\ (=\rho_{eq})} + \sum_{m\neq m'} c_m c_{m'}^* e^{-i(E_m - E_{m'})t} |m\rangle\langle m'|$$

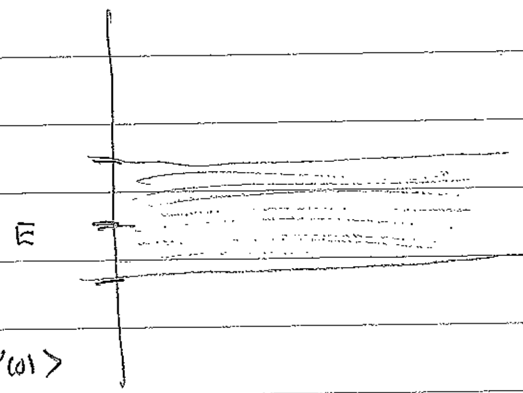


$\sigma_\mathcal{O}(t) = |\mathrm{Tr}(\rho(t)\mathcal{O}) - \mathrm{Tr}(\rho_{DIAG}\,\mathcal{O})|$ unobservably small or
exceedingly rare on realistic time scales

$\Downarrow$

"EQUILIBRATION" Reimann (2008)

WHAT ABOUT THERMALIZATION ?



} ΔE experimentally unresolvable energy interval

$E = \langle\Psi(0)|H|\Psi(0)\rangle$ = "microcanonical window"

quench Hamiltonian

Assume that the microcanonical ensemble contains only states with energies $\in [E-\Delta E/2, E+\Delta E/2]$.
# of levels $\approx D$

$$\rho_{mc} = \frac{1}{D} \sum_{n} |n\rangle\langle n|$$

$|E_n - E| < \Delta E/2$

THERMALIZATION

$$\overline{Tr(\rho_{DIAG}\, O)} = \overline{Tr(\rho_{mc}\, O)}$$

$$= \frac{1}{D} \sum_n O_{nn}$$

$c_n = \langle n | \Psi(0) \rangle$

INITIAL STATE

$$= \sum_n (\rho_{DIAG})_{nn} O_{nn} = \sum_n (\rho_{DIAG})_{nn} O_{nn} = \sum_n |c_n|^2 O_{nn}$$

$|E_n - E| < \Delta E/2$

$$\Downarrow$$

$$\sum_{\substack{n \\ |E_n - E| < \Delta E/2}} |c_n|^2 O_{nn} \approx \frac{1}{D} \sum_{\substack{n \\ |E_n - E| < \Delta E/2}} O_{nn}$$

ALTERNATIVE 1

$|c_n|^2 \approx \frac{1}{D}$

$|c_n|^2$ practically don't fluctuate between eigenstates close in energy

ALTERNATIVE 2

$O_{nn} \approx \text{CONST} = \overline{O}$

O_{nn} practically don't fluctuate between eigenstates close in energy

TO PROVE THIS: USE RANDOM MATRIX THEORY (DEUTSCH, 1991)

SLIDES (Discuss von Neumann, effective temperature T_{eff}, local observables, subsystem, canonical ensemble, ...)

$$\Downarrow$$

$$\sum_n |c_n|^2 O_{nn} \approx \overline{O} \underbrace{\sum_n |c_n|^2}_{=1} = \overline{O}$$

Shrink the microcanonical window to contain only one state: If this state is populated, $|c_n|^2 = 1$, it serves as a thermal state!

Random matrix theory does confirm that $\Theta_{nn} \simeq \text{const}$ for "generic" Hamiltonians (not integrable, not many-body localized, ...)

Sample the Hamiltonian from an ensemble of random matrices which have the same characteristics ("banded", sparse, ...) as the "true" Hamiltonian. Discuss!

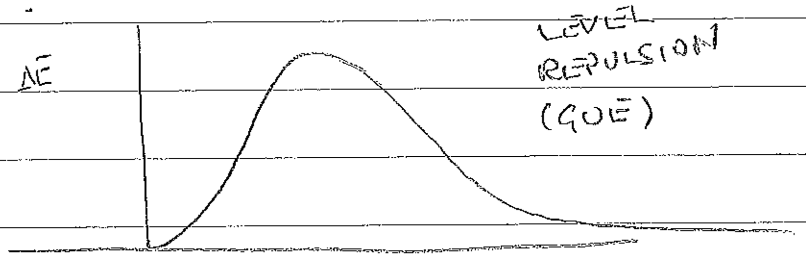
disordered systems, quantum chaos, quantum optics, quantum mesoscopic physics, QCD, Anderson localization, ... and much more

lots of applications of random matrix theory in physics, incl. the ten-fold way classification of symmetry-protected topological phases!

Historic note: E. Wigner, 1930s (?), study of complex nuclear spectra of heavy atoms: level spacing depend only on the symmetry class to which the theory belongs ("Gaussian orthogonal ensemble", "Gaussian unitary", "Gaussian symplectic", ...)

real — Hermitian — self-dual

Statistical properties of the eigenvalues of a large random matrix?



NOW TO SOMETHING ELSE...

PERIODICALLY DRIVEN SYSTEM : FLOQUET DYNAMICS

SLIDE : KAPITZA PENDULUM
from YOUTUBE

Analogously to the Kapitza pendulum, a QUANTUM system may exhibit stationary phases under periodic driving not possible at equilibrium, and even exhibit completely novel phases! Particular interest in the last few years: "Floquet topological engineering" (=> creating nontrivial topological phases that do not exist in equilibrium.
This a very busy and exciting field of research!
For now, today, I'll just present to you the basic formalism necessary to to think about these problems:

Roughly speaking:
Bloch theory in time!

FLOQUET THEORY (Gaston Floquet, 1881-83)

A quantum system described by a time-periodic Hamiltonian

$$H(t) = H(t+\underbrace{T}_{\text{period}}) \qquad (\text{II}.15)$$

posesses states $|\Psi_n(t)\rangle$ (called "Floquet states") which are solutions to the time-periodic Schrödinger equation

$$H(t)|\Psi(t)\rangle \overset{(\hbar=1)}{=} i\partial_t|\Psi(t)\rangle \qquad (\text{II}.16)$$

and have the form

$$|\Psi_n(t)\rangle = e^{-i\overset{\text{QUASIENERGIES}}{\varepsilon_n} t}\underbrace{|u_n(t)\rangle}_{\text{FLOQUET MODES}} \qquad (\text{II}.17)$$

$$|u_n(t)\rangle = |u_n(t+T)\rangle$$

The Floquet states $|\Psi_n(t)\rangle$ are eigenstates of the time-evolution operator

time-ordering

$$U(t_2, t_1) = \tilde{T} \exp\left(-i \int_{t_1}^{t_2} H(t)\, dt\right) \qquad (\text{II}.18)$$

over one driving period T :

$$U(t_0+T, t_0)|\Psi_n(t_0)\rangle = e^{-i\varepsilon_n T} |\Psi_n(t_0)\rangle \qquad (\text{II}.19)$$

the eigenvalue $e^{-i\varepsilon_n t}$ is independent of t_0

Let's prove (II.19) and (II.17) !

Consider the eigenvalue problem

orthonormal

$$U(t+T, t)|\Psi_n(t)\rangle = a_n(t,T)|\Psi_n(t)\rangle \qquad (\text{II}.20)$$

$|a_n(t,T)| = 1$ since U is unitary

$a_n(t,T)$ is independent of t. To see how, multiply (II.20) by $U(t'+T, t+T)$ and use that $U(t'+T, t+T)U(t+T,t) = U(t'+T, t)$ and $U(t'+T, t+T) = U(t',t)$:

$$U(t'+T,t)|\Psi_n(t)\rangle = a_n(t,T)U(t',t)|\Psi_n(t)\rangle$$

$$\Rightarrow |\Psi_n(t'+T)\rangle = a_n(t,T)|\Psi_n(t')\rangle$$

$$\Rightarrow U(t'+T, t')|\Psi_n(t')\rangle = a_n(t,T)|\Psi_n(t')\rangle$$

$t \to t'$ in (II.20)

$$\Rightarrow a_n(t',T) = a_n(t,T) = a_n(T) \qquad (\text{II}.21)$$

It follows from (II.20) and (II.21) that we can express

$$a_n(T) = e^{-i\varepsilon_n T}, \quad \varepsilon_n \in \mathbb{R} \qquad (\text{II}.22)$$

It follows from (II.20) and (II.22) that

$$|\Psi_n(t+T)\rangle = e^{-i\varepsilon_n T}|\Psi_n(t)\rangle \qquad (\text{II}.23)$$

which can be written as $|\Psi_n(t)\rangle = e^{-i\varepsilon_n t}|u_n(t)\rangle$ with

$$|u_n(t)\rangle = |u_n(t+T)\rangle \qquad (\text{II}.24)$$

which proves (II.17).

We can expand $U(t_2,t_1)$ in the basis of Floquet states, yielding (N.B. U acts as a time evolution operator on the Floquet states $|\Psi_n(t)\rangle$, hence the phases in (II.25)):

$$U(t_2,t_1) = \sum_n e^{-i\varepsilon_n(t_2-t_1)}|u_n(t_2)\rangle\langle u_n(t_1)| \qquad (\text{II}.25)$$

It follows that *

$c_n = \langle u_n(t_0)|\Psi(t_0)\rangle$

$$|\Psi(t)\rangle = \sum_n c_n e^{-i\varepsilon_n(t-t_0)}|u_n(t)\rangle \qquad (\text{II}.26)$$

(II.26) implies (IMPORTANT!)

- If the system is prepared in a single Floquet state, $|c_n| = \delta_{n,n_0}$, its time evolution will be periodic and described by the Floquet mode $|u_{n_0}(t)\rangle$ (up to an irrelevant overall phase factor)

$$* \; |\Psi(t)\rangle = U(t,t_0)|\Psi(t_0)\rangle \overset{(\text{II}.25)}{=} \sum_n e^{-i\varepsilon_n(t-t_0)}|u_n(t)\rangle \overbrace{\langle u_n(t_0)|\Psi(t_0)\rangle}^{c_n}$$

- On the other hand, if the system is prepared in a coherent superposition of several Floquet states, the time evolution will be determined by two contributions

"MICROMOTION" → the periodic time ~~dependence of the~~ Floquet modes $|u_n(t)\rangle$

relative dephasing ~~of the factors~~ $e^{-i\varepsilon_n(t-t_0)}$ in (II.26)
(like in ordinary time evolution of a ~~time-independent~~ Hamiltonian, with the quasi energies ε_n ~~playing the role of~~ energy!)

Suppose that we want to study our periodically driven system over a time span that is $\gg T$ (period of the drive). We can then ignore the micromotion and study the time evolution STROBOSCOPICALLY, i.e. in steps of the driving period T. The stroboscopic time evolution is described by the TIME-INDEPENDENT FLOQUET HAMILTONIAN $H^F_{t_0}$ which generates the time evolution over one period

$$\mathcal{U}(t_0+mT, t_0) = \exp\left(-iH^F_{t_0} mT\right), \quad m\in\mathbb{Z} \qquad (\text{II}.27)$$

where $H^F_{t_0}$ can be expressed as

$$H^F_{t_0} = \sum_n \varepsilon_n |u_n(t_0)\rangle\langle u_n(t_0)| \qquad (\text{II}.28)$$

t_0: reference time (a constant)

* $\underbrace{|u_n(t_0+T)\rangle}_{e^{i\varepsilon_n(t_0+T)}|\psi_n(t_0+T)\rangle} = |u_n(t_0)\rangle$

$= \mathcal{U}(t_0+T,t_0)\, e^{i\varepsilon_n(t_0+T)}\underbrace{|\psi_n(t_0)\rangle}_{e^{-i\varepsilon_n t_0}|u_n(t_0)\rangle}$

$\Rightarrow \mathcal{U}(t_0+T,t_0)|u_n(t_0)\rangle = e^{-i\varepsilon_n T}|u_n(t_0)\rangle$

Q.E.D.

proof: Taylor expand $\exp(-iH^F_{t_0}t)$, use ON property of the Floquet modes, apply to $|u_n(t_0)\rangle$ and read off

* $\mathcal{U}(t_0+mT,t_0)|u_n(t_0)\rangle = e^{-i\varepsilon_n mT}|u_n(t_0)\rangle$

Q.E.D.

Let's pause and Eq. (II.19): Let $\Omega = \frac{2\pi}{T}$ be the frequency of the drive. We see immediately that a shift $\mathcal{E}_n \to \mathcal{E}_n + \Omega$ leaves the eigenvalues in (II.19) invariant. In other words, the quasienergies $\mathcal{E}_n$ are periodic with period $\Omega = \frac{2\pi}{T}$.

SLIDES Discuss analogy with Bloch and quasimomenta $\longrightarrow$ FLOQUET ZONE (a.k.a. "Brillouin zone" in time!)

Moreover, from

$$\exp\left(-iH^F_{t_0} mT\right)|u_n(t_0)\rangle = \exp\left(-i\mathcal{E}_n mT\right)|u_n(t_0)\rangle$$

it follows that

$$H^F_{t_0} |u_n(t_0)\rangle = \mathcal{E}_n |u_n(t_0)\rangle \qquad (\text{II}.29)$$

with $H^F_{t_0+T} = H^F_{t_0} = \mathcal{U}^{-1}(t_0', t_0) H^F_{t_0'} \mathcal{U}(t_0', t_0)$

Let's put the system on a lattice, and Fourier transform to reciprocal space. We may then write (II.29) as

assume 1D

$$H^F_{t_0}(k)\,|u_n(k,t_0)\rangle = \mathcal{E}_n(k)\,|u_n(k,t_0)\rangle \qquad (\text{II}.30)$$

Let's choose, say, $t_0 = 0$, and suppress this parameter. Then,

$$H^F(k)\,|u_n(k)\rangle = \mathcal{E}_n(k)\,|u_n(k)\rangle \qquad (\text{II}.31)$$

This looks like an eigenvalue problem for a Bloch Hamiltonian with cell-periodic Bloch states $|u_n(k)\rangle$ and energy bands $\mathcal{E}_n(k)$!!!

Having established that the STROBOSCOPIC PROBLEM (observation time >> drive period) is effectively a time-independent problem coded by some FLOQUET HAMILTONIAN, how do we actually obtain this effective Hamiltonian? Hard problem! Very much an active research topic! Why? By cleverly choosing some system (described by a time-independent Hamiltonian) and then cleverly choosing a DRIVE PROTOCOL (periodic driving from some external field, e.g. a pulsed laser) one can manufacture Floquet Hamiltonians for the stroboscopic problem with cool properties.

↯ Much work these days:
FLOQUET TOPOLOGICAL ENGINEERING:
engineering novel topological phases.

Bad news: Really hard problem to get out a Floquet Hamiltonian. Reliable perturbative techniques only for high-frequency drives (Ω >> characteristic energy of the system being driven)

Still, lot of progress has been made. For example, the Haldane model for a Chern insulator has been realized via a Floquet Hamiltonian approach (shining lasers on cold atoms in an optical lattice ... more about this later...)