TOPOLOGICAL SUPERCONDUCTOR IN 1D: KITAEV CHAIN

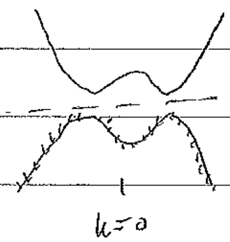
Last lecture I introduced the Hamiltonian for a spinless p-wave superconductor in 1D and showed that we could write it on the form

NAMBU SPINOR $\begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix}$

$$H = \frac{1}{2}\sum_k \underbrace{\Psi_k^+}_{\left(c_k^+ \; c_{-k}\right)} \underbrace{\begin{pmatrix} \left(\frac{k^2}{2m}-\mu\right) & \Delta_0 k \\ \Delta_0^* k & -\frac{k^2}{2m}+\mu \end{pmatrix}}_{H(k)\text{. BdG single-particle Hamiltonian}} \Psi_k$$

$$E_\pm(k) = \pm\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2 + |\Delta_0|^2 k^2}$$

$E_-(k)$ band completely filled $\rightarrow$ looks like a band insulator



The band structure looks particle-hole symmetric. Check!

Note from the figure on the previous page that H_{BdG} has PARTICLE-HOLE SYMMETRY, a "built-in" feature of the Bogoliubov-de Gennes formalism. Explicitly, directly on the Hamiltonian $H(k)$ in (III.57):

$$\underbrace{(\sigma_x K)}_{C}\, H(k)\, (\sigma_x K)^{-1} = -H(-k) \qquad (\text{III}.59)$$

C (particle-hole transformation)

Any anti-unitary transformation UK with the property that $\underbrace{(UK)H(k)(UK)^{-1} = -H(-k)}$ will do

The spectrum is gapped as long as $\mu \neq 0$. $\mu > 0$ (<0) is topologically nontrivial (trivial) as we shall see below

Write $\Delta_0 = e^{i\varphi}|\Delta_0|$ and absorb the phase in the c_k-operators in (III.55): $c_k^+ \to e^{i\varphi/2}c_k^+$, $c_k \to e^{-i\varphi/2}c_k$. Taking the limit $m \to \infty$, $H(k)$ then reduces to the 1D Dirac Hamiltonian

$$H(k) \xrightarrow{m\to\infty} |\Delta_0| \begin{pmatrix} -\mu/|\Delta_0| & k \\ k & \mu/|\Delta_0| \end{pmatrix} \qquad (\text{III}.60)$$

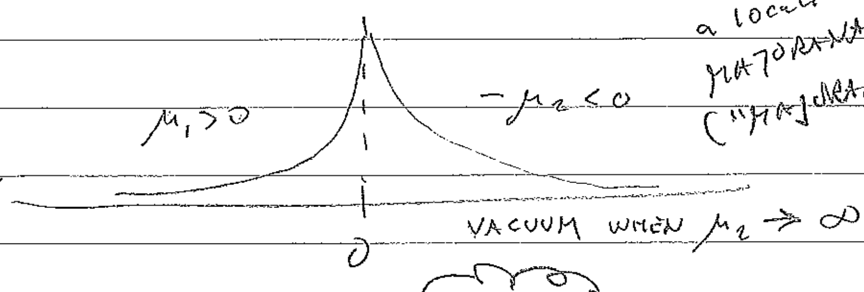
SAME ANALYSIS AS FOR THE SSH MODEL, p. I42

Taking $\mu = \mu(x)$, with a profile $\mu(x) = \begin{cases} \mu_1, & x < 0 \\ -\mu_2, & x > 0 \end{cases}$ $\quad \mu_1, \mu_2 > 0$

one finds the zero-energy solution

$$\Psi(x) \sim \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-|\mu(x)\, x|} \qquad (\text{III}.61)$$

One can argue that this state describes a localized zero-energy MAJORANA FERMION ("MAJORANA ZERO-MODE") (*)



(*) The difference from SSH is because of the Nambu spinor structure of the present problem

see book by Bernevig, p 198f

SLIDES

To explicitly see how this comes about, it's advantageous to put the 1D p-wave superconductor on a lattice. This takes us to the KITAEV CHAIN (A. Kitaev, Phys. Usp. 44, 131 (2001))

$$H = \sum_{j=1}^{N-1}\left[-t\left(c_j^{+}c_{j+1} + c_{j+1}^{+}c_j\right) + |\Delta_0|\left(c_{j+1}c_j^{+} + c_j c_{j+1}\right)\right] - \mu\sum_{j=1}^{N} c_j^{+}c_j$$

$t > 0$

$$\Psi_k^{+} = (c_k^{+}\ c_{-k}), \quad \Psi_k = \begin{pmatrix} c_k \\ c_{-k}^{+} \end{pmatrix} \qquad (\text{III}.62)$$

$\Downarrow$ Fourier transformation, $H = \sum_k \Psi_{-k}^{+} H(k) \Psi_k$

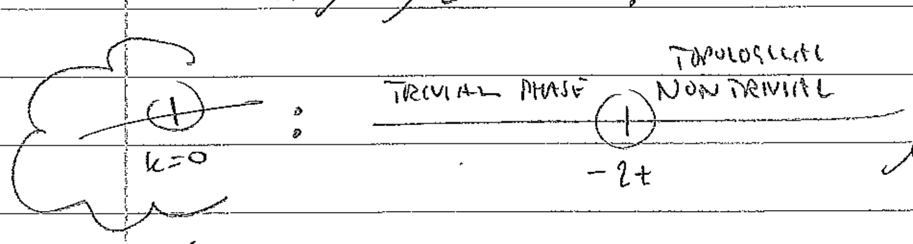
$$H_{BdG} = H(k) = \begin{pmatrix} -2t\cos k - \mu & 2i|\Delta_0|\sin k \\ -2i|\Delta_0|\sin k & 2t\cos k + \mu \end{pmatrix} \qquad (\text{III}.63)$$

$\Downarrow$ diagonalization

$$E_{\pm}(k) = \pm\sqrt{(2t\cos k + \mu)^2 + 4|\Delta_0|^2\sin^2 k} \qquad (\text{III}.64)$$

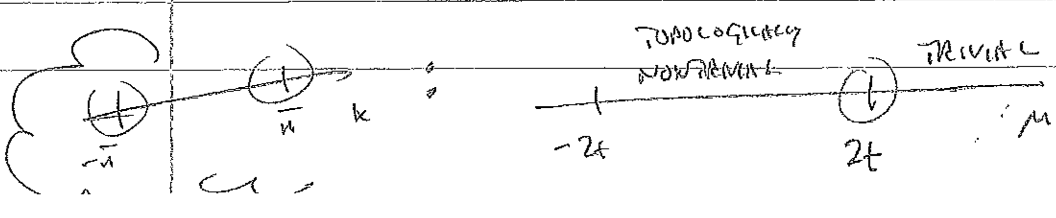
$\xrightarrow{k \to 0}$ (III.58) ($k = 0 \Leftrightarrow$ LONG WAVELENGTH LIMIT $\Leftrightarrow$ CONTINUUM LIMIT)

For $|\Delta_0| \neq 0$ (superconductor!) the gap closes at $k = 0$ when $\mu = \mu_c = -2t$.



(This is consistent with the continuum limit with $m \sim \frac{1}{t} \to \infty$.)

N.B. The gap can also close at the BZ boundaries



We don't consider this case here

Next, let' split the electron operators c_j, c_j^+ into their real - and imaginary parts

odd index for real part ↓ even index for imaginary part

$$\begin{cases} c_j = \frac{1}{2}(\gamma_{j,1} + i\gamma_{j,2}) \equiv \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j}) \\ c_j^+ = \frac{1}{2}(\gamma_{j,1} - i\gamma_{j,2}) \equiv \frac{1}{2}(\gamma_{2j-1} - i\gamma_{2j}) \end{cases} \qquad (\text{III}.65)$$

with $\gamma_{j,1} \equiv \gamma_{2j-1}$, $\gamma_{j,2} \equiv \gamma_{2j}$ associated with site j. Easy to prove that they are Hermitian, hence Majoranas : $\gamma_{2j-1} \equiv (c_j^+ + c_j)$, $\gamma_{2j} \equiv i(c_j^+ - c_j)$ from which it follows :

$$\begin{cases} \{\gamma_i^+, \gamma_{i'}\} = 2\delta_{ii'} \\ \{\gamma_i, \gamma_{i'}\} = 2\delta_{ii'} \quad \Rightarrow \gamma_i^2 = 1 \end{cases} \qquad (\text{III}.66)$$

number operator ?

$$\gamma_i^+\gamma_i = \gamma_i\gamma_i^+ = 1$$

⇓

We can't use the Majorana operators to "count" the number of Majoranas in a state. A Majorana state is in some sense always empty and always filled. Only [pairs of Majoranas] = = [complex fermions] can be properly accounted for.

Using the Majorana representation, the Kitaev Hamiltonian in (III.62) becomes

$$H = \frac{i}{2}\sum_j \left(-\mu\,\gamma_{2j-1}\gamma_{2j} + (t+|\Delta_0|)\,\gamma_{2j}\gamma_{2j+1} + (-t+|\Delta_0|)\,\gamma_{2j-1}\gamma_{2j+2}\right) \qquad (III.67)$$

Let's consider two limiting cases of H:

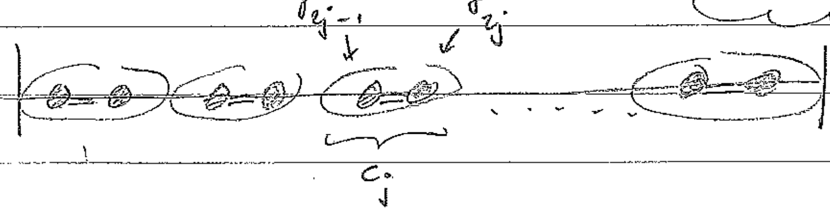
(1) $|\Delta_0| = t = 0$, $\mu < 0$

$$\Downarrow$$

$$H = -\mu\frac{i}{2}\sum_{j=1}^{N}\gamma_{2j-1}\gamma_{2j} \qquad (III.68)$$

In this phase the Majorana operators on each lattice site are coupled, but Majorana operators between sites are decoupled. Pictorially:

TRIVIAL PHASE



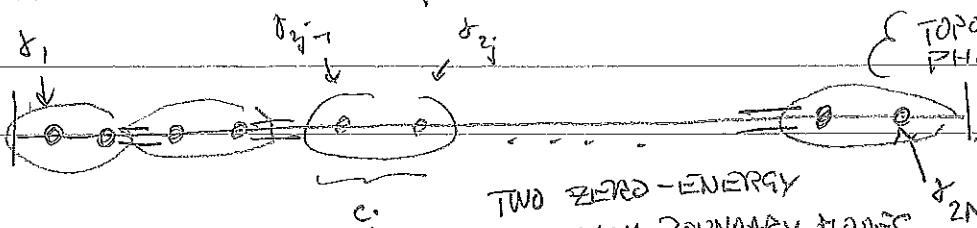
(2) $|\Delta_0| = t > 0$, $\mu = 0$

$$\Downarrow$$

$$H = it\sum_{j=1}^{N-1}\gamma_{2j}\gamma_{2j+1} \qquad (III.69).$$

The Majorana operators γ_1 and γ_{2N} at the end points of the chain are not coupled to the rest of the chain!

TOPOLOGICAL PHASE



TWO ZERO-ENERGY MAJORANA BOUNDARY MODES

We can combine the two boundary Majorana operators to a single NonLocal (complex) fermion operator

$$c_{\substack{\text{NON}\\\text{LOCAL}}} = \frac{1}{2}(\gamma_0 + i\gamma_{2N}) \qquad (\text{III}.70)$$

Since this operator is absent from the Hamiltonian (III.69) the states $|GS\rangle$ and $c^{+}_{\substack{\text{NON}\\\text{LOCAL}}}|GS\rangle$ are degenerate

$|GS\rangle$: ground state with no NL fermion

$c^{+}_{\substack{\text{NON}\\\text{LOCAL}}}|GS\rangle$: ground state with NL fermion

$\Downarrow$

topologically nontrivial superconducting ground state is two-fold degenerate

with 0 or 1 NonLocal fermion !

In the topologically nontrivial phase, $|\mu| < 2t$, the wave functions of the Majorana end modes are exponentially localized ⇒ robust zero-energy modes for sufficiently large chains so that there is no hybridization between these modes.

What about symmetry class? For the case we have studied, with all parameters real, we can put the Kitaev chain in symmetry class BDI (T, C, and S symmetries OK) if we require any local perturbation that does not close the gap to respect these symmetries. By going to a basis where $H(k)$ is off-diagonal we can calculate the winding number ν in the same way as for the SSH chain to find $\nu = 0$ (1)) for the topologically trivial (nontrivial) phase with 0 (1) Majorana modes / edge.

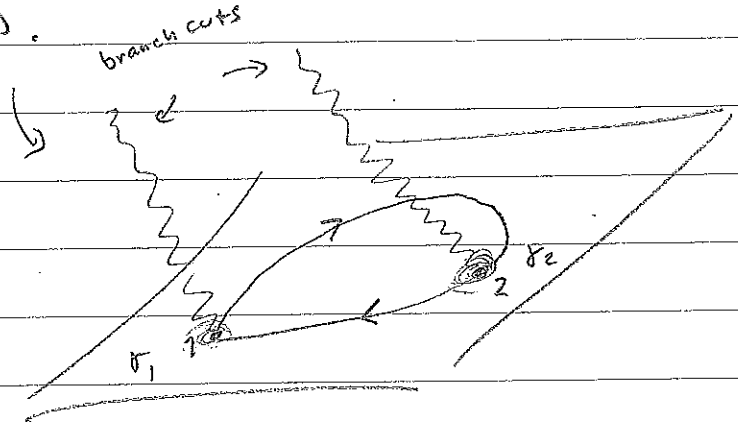
If T is broken (by allowing complex parameters in H), the symmetry class is usually chosen as D, with a $\mathbb{Z}_2$ topological index.

The Majorana zero-energy modes obey NON ABELIAN STATISTICS and for this reason may be used to store and manipulate QUANTUM INFORMATION.

$\rightarrow$

NON ABELIAN STATISTICS OF MAJORANA ZERO MODES

How do we exchange two spinless fermions in 1D without violating Pauli? We need one more dimension to take one of them around the other! → 2D, for example a network of 1D chains (seriously, this has been proposed! See slides later) or a 2D p-wave superconductor, type II, where the magnetic vortices host unpaired Majoranas. For now, let's not worry about the specific set-up, but simply assume that we are in 2D.



For a starter: let's exchange two Majoranas, γ_1 and γ_2. Since we're dealing with a superconductor with (in general) a complex order parameter

$$\Delta \sim e^{i\theta}|\Delta| \qquad (\text{III}.70)$$

we need to set things up in such a way that Δ is single-valued as we move around on a loop in the plane (like in the picture above). This can be achieved by slicing up the complex plane with branch cuts. We choose to draw these branch cuts in such a way so that one of the Majoranas crosses one branch cut, by which it picks up a phase π.

(This all sounds quite arbitrary and "made up" but can be put on firm ground by considering vortices in a 2D p-wave superconductor D.A. Ivanov, Phys. Rev. Lett. 86, 268 (2001).) DISCUSS!

With this, we obtain

$$\gamma_1 \to e^{i\pi}\gamma_2 = -\gamma_2 \qquad (\text{III}.71)$$
$$\gamma_2 \to \gamma_1$$

We can implement this exchange using the BRAID OPERATOR $B_{12} \equiv \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)$:

$$\gamma_i \to B_{12}\gamma_i B_{12}^{+} \qquad (\text{III}.72)$$

A double exchange (bringing γ_i back to its initial position i, topologically equivalent with taking one Majorana around the other) is then described by

$$\gamma_i \to B_{12}B_{12}\gamma_i B_{12}^{+}B_{12}^{+} = (\gamma_1\gamma_2)\gamma_i\underbrace{(\gamma_1\gamma_2)^{+}}_{\gamma_2\gamma_1} = -\gamma_i \qquad (\text{III}.73)$$

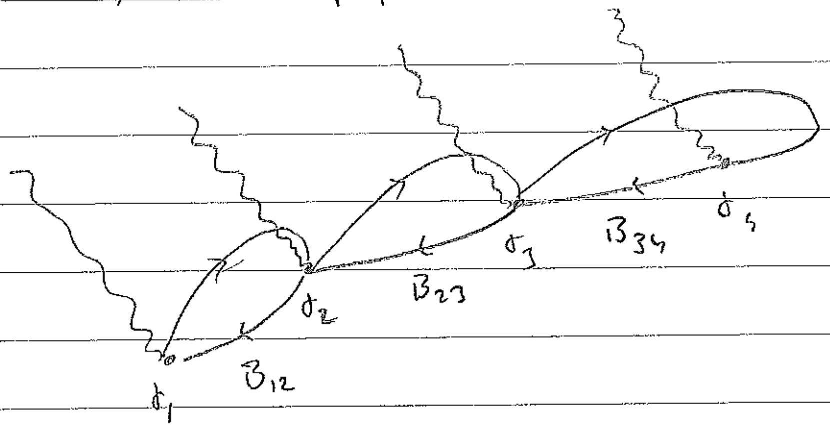
Let's explore how the braid operation acts on the states $|0\rangle$ and $|1\rangle = c_1^{+}|0\rangle = \frac{1}{2}(\gamma_1 - i\gamma_2)|0\rangle$

empty — one (spinless) electron

Remember: When dealing with number states we always have to go back to the electron operators, using that $\gamma_1 = (c_1^{+}+c_1)$, $\gamma_2 = i(c_1^{+}-c_1)$. Plugging this into the expression for B_{12}, using that $c_1^{+}|0\rangle = |1\rangle$, etc. one finds that

$$\begin{cases} B_{12}|0\rangle = \frac{1}{\sqrt{2}}(1+i)|0\rangle \\ B_{12}|1\rangle = \frac{1}{\sqrt{2}}(1-i)|1\rangle \end{cases} \qquad (\text{III}.74)$$

To find nontrivial effects of braiding on the physical electron states we must allow for at least two electrons. (This will also prepare us for going from "physical" to "logical" qubits, necessary for doing quantum information tasks!)



$$\begin{cases} c = (\gamma_1 + i\gamma_2)/2 \Rightarrow \gamma_1 = (c^+ + c), \ \gamma_2 = i(c^+ - c) \\ d = (\gamma_3 + i\gamma_4)/2 \Rightarrow \gamma_3 = (d^+ + d), \ \gamma_4 = i(d^+ - d) \end{cases}$$

$c^+|00\rangle = |10\rangle, \ d^+|00\rangle = |01\rangle, \ c^+d^+|00\rangle = |11\rangle, \dots$ etc

$B_{12} = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)$

$$\rightarrow B_{12}|00\rangle = \frac{1}{\sqrt{2}}\left(1+(c^+ + c)i(c^+ - \cancel{c})\right)|00\rangle$$
$$= \frac{1}{\sqrt{2}}\left(1 + \cancel{c^+ i c^+} + c\, i c^+\right)|00\rangle$$
$$= \frac{1}{\sqrt{2}}(1+i)|00\rangle$$

$B_{23} = \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_3)$

$$\rightarrow B_{23}|00\rangle = \frac{1}{\sqrt{2}}\left(1+i(c^+ - c)(d^+ + \cancel{d})\right)|00\rangle$$
$$= \frac{1}{\sqrt{2}}\left(|00\rangle + i|11\rangle\right)$$

$\vdots$

$$B_{34}|00\rangle = \frac{1}{\sqrt{2}}(1+i)|00\rangle$$

The NON-ABELIAN exchange statistics is now apparent: Whenever two exchanges involve some of the same Majoranas the braid operators do not commute

$$\left([B_{i-1,i}, B_{i+1,i}] = \gamma_{i-1}\gamma_{i+1} \right)$$

Introduce "LOGICAL QUBITS" (spanning a "COMPUTATIONAL BASIS")

$$|\bar{0}\rangle = |00\rangle \quad , \quad |\bar{1}\rangle = |11\rangle$$

In the $\{|\bar{0}\rangle, |\bar{1}\rangle\}$ basis the Pauli matrices can be represented in terms of products of Majorana operators

$$\begin{cases} \sigma_z = -i\gamma_1\gamma_2 = -i\gamma_3\gamma_4 \\ \sigma_x = -i\gamma_2\gamma_3 \\ \sigma_y = -i\gamma_1\gamma_3 = -i\gamma_2\gamma_4 \end{cases}$$

We can use this to represent SINGLE-QUBIT QUANTUM GATES by BRAIDING OF MAJORANAS!

EX NOT GATE: $|\bar{0}\rangle \xrightarrow{\text{NOT}} |\bar{1}\rangle \quad , \quad |\bar{1}\rangle \xrightarrow{\text{NOT}} |\bar{0}\rangle$

$$\underset{B_{23}B_{23}}{\Updownarrow} \qquad \begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$$

$$B_{23}B_{23} = \frac{1}{\sqrt{2}}(1+\gamma_{23})\frac{1}{\sqrt{2}}(1+\gamma_{23}) = \frac{1}{2}(1+i\sigma_x)(1+i\sigma_x)$$

$$= \frac{1}{2}\left(1+\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\right)\left(1+\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\sigma^x$$

$i\sigma^x\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix}$, $i\sigma^x\begin{pmatrix}0\\1\end{pmatrix} = i\begin{pmatrix}1\\0\end{pmatrix}$ $\Rightarrow$ NOT (up to a phase)

Problem?

acting on
$|a\rangle \otimes |b\rangle = v_{00}|\bar{0}\bar{0}\rangle + v_{01}|\bar{0}\bar{1}\rangle + v_{10}|\bar{1}\bar{0}\rangle + v_{11}|\bar{1}\bar{1}\rangle$

"Universal" set of quantum gates:

π/8 gate

$$\underbrace{R_{[\pi/4]} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}}_{?}, \quad \underbrace{H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\text{can be realized through braiding of Majoranas}}, \quad \underbrace{\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\text{cannot!}}$$

Hadamard

phase gate

$$\underbrace{S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}}_{?}$$

$$|\bar{0}\rangle \to \frac{1}{\sqrt{2}}(|\bar{0}\rangle + |\bar{1}\rangle)$$
$$|\bar{1}\rangle \to \frac{1}{\sqrt{2}}(|\bar{0}\rangle - |\bar{1}\rangle)$$

Still, Majorana-based logical qubits can be used to store quantum information. DISCUSS! Measurement-based protocols, etc. Braiding is problematic because of "diabatic" errors (large!), etc.

How realistic is all this? How to produce Majoranas in the lab? And how to manipulate them?!

SLIDES