THE QUANTUM SPIN HALL INSULATOR
(A.K.A. "2D TOPOLOGICAL INSULATOR")

IQHE → Chern insulator → QSH

Kane-Mele 2005-06
Bernevig et al 2006
Molenkamp et al 2007

Moore-Balents picture of the Z_2 invariant.

Extensions, ...

3D topological insulators

SLIDES

TOPOLOGICAL SUPERCONDUCTIVITY:
SOME BACKGROUND

When discussing the "ten-fold way" (periodic table of symmetry-protected topological systems) I stressed that the classification applies to band insulators AND (mean-field = non-interacting superconductors. We have discussed four models of topological band insulators (or classes of models): SSH, IQHE, Chern insulators, and quantum spin Hall systems. What about the superconductors? Fascinating subject! Here we'll only touch upon it, via a case study of the very simplest model THE KITAEV CHAIN.

noninteracting fermionic

As a warm-up, let's recall some basics of ordinary s-wave superconductivity, in the language of BCS.

Starting with the Fröhlich Hamiltonian (keeping only quartic phonon-mediated (effective) electron-electron interaction terms after the Schrieffer-Wolff transformation):

$(\hbar=1)$

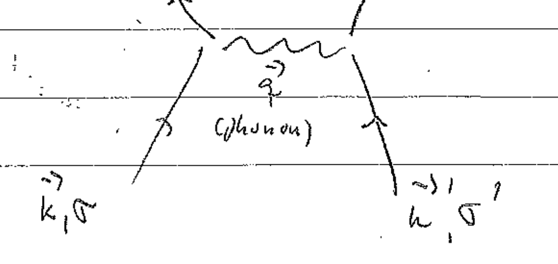
$$H = \sum_{\vec{k},\sigma=\uparrow,\downarrow} c^{\dagger}_{\vec{k}\sigma}\left(\frac{k^2}{2m}-\mu\right)c_{\vec{k}\sigma}$$

$$+\sum_{\vec{k},\vec{k}',\vec{q},\sigma,\sigma'} V_{\vec{k}\vec{k}'\vec{q}}\, c^{\dagger}_{\vec{k}'+\vec{q},\sigma'}\, c^{\dagger}_{\vec{k}-\vec{q},\sigma}\, c_{\vec{k},\sigma}\, c_{\vec{k}',\sigma'}$$

$$+\cancel{H_{ph}} + \cancel{H_{el-el}}$$

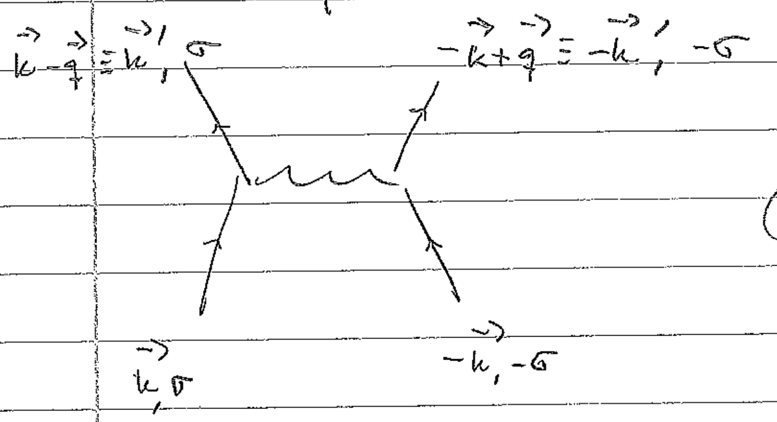
phonons   coulomb

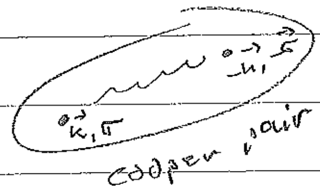
(III.46)



Cooper : important interaction terms are those with
$\vec{k}' = -\vec{k}, \ \sigma' = -\sigma \ \Rightarrow$ "Cooper instability", formation of Cooper pairs

Relabelling, we then obtain the interaction term, call it H_{int}





$$H_{int} = \sum_{\substack{\vec{k},\vec{k}' \\ \sigma,\sigma'}} \underbrace{V_{\vec{k},\vec{k}'}}_{\approx -V = \text{CONSTANT}} c^{\dagger}_{\vec{k}',\sigma'} c^{\dagger}_{-\vec{k}',-\sigma'} c_{-\vec{k},-\sigma} c_{\vec{k},\sigma} \qquad (\text{III}.47)$$

Bardeen - Cooper - Schrieffer (1957) : Consider the ground state wave function as representing a grand canonical ensemble *)

$$|\Psi_{BCS}\rangle = \prod_{\vec{k}} \left(u_{\vec{k}} + v_{\vec{k}} c^{\dagger}_{\vec{k},\uparrow} c^{\dagger}_{-\vec{k},\downarrow} \right) \overset{\text{vacuum}}{|0\rangle} \qquad (\text{III}.48)$$

i.e. a superposition of states with different number of Cooper pairs

$$\Downarrow \quad \text{NONZERO ANOMALOUS EXPECTATION VALUES POSSIBLE!}$$

$$\underbrace{\langle c_{-\vec{k},-\sigma} c_{\vec{k},\sigma} \rangle}_{{}_{BCS}\langle \ldots \rangle_{BCS}} \neq 0 \qquad (\text{III}.49)$$

*) to be historically correct :
this is actually an afterthought !

Now consider $\langle c_{-\vec{k},-\sigma} c_{\vec{k},\sigma} \rangle$ to be a mean-field approximation for $c_{-\vec{k},-\sigma} c_{\vec{k},\sigma}$ (and similarly for $\langle c^{+}_{\vec{k}',\sigma'} c^{+}_{-\vec{k}',-\sigma'} \rangle$), assuming that the variation around the mean value is small. "Insertion"* into (III.47) then yields the BCS HAMILTONIAN (adding the kinetic term from (III.46)):

$$H_{BCS} = \sum_{\vec{k},\sigma} c^{+}_{\vec{k}\sigma} \left(\frac{k^2}{2m} - \mu \right) c_{\vec{k}\sigma} + \sum_{\vec{k},\sigma} \left(\Delta_{\vec{k}} c^{+}_{\vec{k}\sigma} c^{+}_{-\vec{k},-\sigma} + \Delta^{*}_{\vec{k}} c_{-\vec{k},-\sigma} c_{\vec{k},\sigma} \right) \quad (\text{III}.50)$$

where

BCS ORDER PARAMETER

$$\Delta_{\vec{k}} = -V \langle c^{+}_{\vec{k},\sigma} c^{+}_{-\vec{k},-\sigma} \rangle \quad (\text{III}.51)$$

measures the gain in energy by formation of such a Cooper pair ≈ "binding energy"

[amplitude for removing a Cooper pair in the state $|-\vec{k},-\sigma; \vec{k},\sigma\rangle$ = amplitude for existence of such an occupied state

ELITZUR'S THEOREM: "(LOCAL) GAUGE SYMMETRIES CANNOT BE BROKEN" Phys. Rev. D 12, 3978 (1975)

What is the broken symmetry in the BCS ground state?

Text book answer: U(1) gauge symmetry, $|\Psi_{BCS}\rangle$ is not invariant under $c^{+}_{\vec{k},\uparrow} \to e^{i\varphi(\vec{k})} c^{+}_{\vec{k},\uparrow}$.

But note: This symmetry is already broken in the BCS Hamiltonian ⇒ artifact of the mean-field approximation.

Discuss! Global symmetry can be broken, cf. Anderson-Higgs!

*/ "Add and subtract" $\langle c_{-\vec{k},-\sigma} c_{\vec{k},\sigma} \rangle$ and $\langle c^{+}_{\vec{k}',\sigma'} c^{+}_{-\vec{k}',-\sigma'} \rangle$ in (III.47), expand, and discard terms that are quadratic in the variati[on]

The BCS order parameter codes for S-WAVE SUPERCONDUCTIVITY with the electrons (really: quasi-particles!) in a $\underbrace{\text{singlet spin state } (S=0)}_{\text{antisymmetric}} \otimes \underbrace{\text{S-wave spatial state}}_{\text{symmetric}}$

OK with Pauli!

Suppose we were to put the electrons making up a Cooper pair in, say, a $\underbrace{\text{triplet spin state}}_{\text{symmetric}}$ (S = 1). By Pauli, we then have to put the spatial part of the state in a partial wave with $l = 1, 3, \ldots$

"p-wave", simplest choice!

Singlet Cooper pairing has $\Delta_{\vec{k}} = \Delta_{-\vec{k}}$ (III.52)

Triplet —"— $\Delta_{\vec{k}} = -\Delta_{-\vec{k}}$

A famous candidate material with triplet superconductivity (recently controversial: cf. the talk by Sigrist, MC2, 28/11!), is Sr_2CuO_4 with order parameter (Quasi-2D Fermi liquid in its "normal" state)

$$\Delta_{\vec{k},\alpha,\beta} \sim \langle c_{\vec{k}\alpha} c_{-\vec{k}\beta} \rangle = \text{CONST } (k_x + i k_y)\, \delta_{\alpha,\beta} \quad (\text{III}.53)$$

$\Downarrow$

"$p_x + i p_y$"

N.B. Some of the best experimentalists are playing with this toy!

In the following, we'll look at a simpler (toy!) model for p-wave pairing in 1D, considering spinless (= spin polarized) electrons, and making the choice

$$\Delta_k = \Delta_0 k \quad (\text{III}.54)$$

Δ_0: CONSTANT $\in \mathbb{C}$

($\overline{\text{III}}$ .23

($\hbar=1$)

$$H = \sum_k c_k^\dagger \left(\frac{k^2}{2m} - \mu\right) c_k + \frac{1}{2}\sum_k \left(\Delta_0 k\, c_k^\dagger c_{-k}^\dagger + \Delta_0^* k\, c_{-k} c_k\right) \qquad (\overline{\text{III}}.55)$$

(Although I implicitly assume discrete values of k by doing a summation instead of integration, let's turn now think about the model as a continuum model → no lattice, no BZ, "no nothing!")

We can "conquer" the problem to a first-quantized quadratic Hamiltonian (BOGOLIUBOV - DE GENNES HAMILTONIAN) by introducing the NAMBU SPINOR

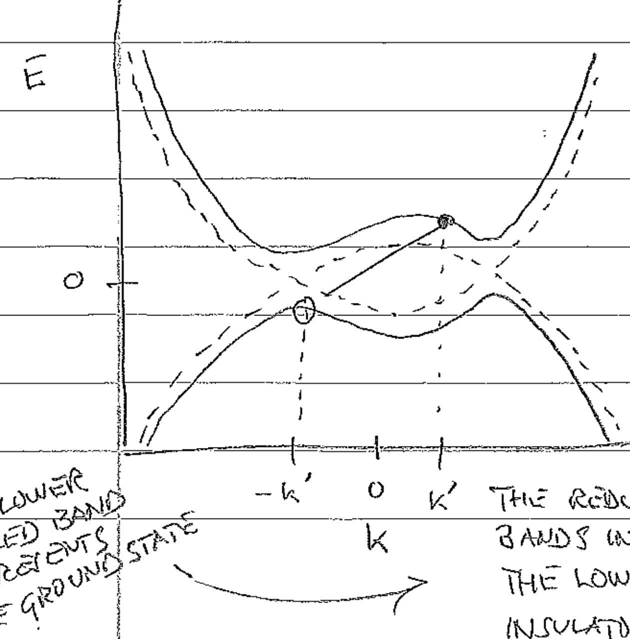
$$\Psi_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \qquad (\overline{\text{III}}.56)$$

and write

$$H = \frac{1}{2}\sum_k \Psi_k^\dagger \underbrace{\begin{pmatrix} \frac{k^2}{2m} - \mu & \Delta_0 k \\ \Delta_0^* k & -\frac{k^2}{2m} + \mu \end{pmatrix}}_{H(k) = H_{BdG}} \Psi_k \qquad (\overline{\text{III}}.57)$$

$\Downarrow$ diagonalization

$$E_\pm = \pm\sqrt{\left(\frac{k^2}{2m} - \mu\right)^2 + |\Delta_0|^2 k^2} \qquad (\overline{\text{III}}.58)$$



- - - - - $\Delta = 0,\ \mu = 0$

——— $\Delta \neq 0,\ \mu \neq 0$

THE LOWER FILLED BAND REPRESENTS THE GROUND STATE

THE REDUNDANCY IN THE BdG FORMALISM (TWO BANDS INSTEAD OF ONE) IS REDUCED BY FILLING UP THE LOWER BAND, TAKING $E_F = 0$ (LIKE IN A 2-BAND INSULATOR) AND THEN CONSIDER ONLY POSITIVE ENERGIES.