Chapter 1
Effective Field Theories for Topological states of Matter

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1.1 Introduction

In school we learn about three states of matter: solids, liquids and gases. Typically these occur at different temperatures so that heating a solid first melts it into a liquid, then evaporates the liquid into a gas. The transition between two such phases of matter does not happen gradually as the temperature changes, but is a drastic event, a *phase transition*, that occur at a precisely defined transition temperature. The classification of matter according to the three states just mentioned is not at all exhaustive. For example, solids can be insulating or conducting and depending on the temperatures they can, or cannot, be magnetic.

Over the years, this picture has become more and more refined, but most of the fundamental ideas governing phases of matter was for a long time based on the work of Lev Landau in the 1930's. However, a few decades ago this completely changed with the advent of *topological phases of matter*. As you will see, these phases are of a fundamentally different nature, and they are the subject of these notes. But before we get there some words on terminology:

Matter denote systems with many degrees of freedom, typically collections of "particles" that can be atoms or electrons, but also quantum mechanical spins or

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quasiparticles such as phonons or magnons. In these notes we shall however only deal with systems composed of fermions, and typically these fermions should be thought of as electrons in a solid.

With a *phase of matter* we mean matter with certain characteristic properties such as rigidity, superfluid density, or magnetization. A phase is however not defined by any specific values for these quantities, but by whether they are at all present—for example, the transition between a solid and a liquid happens when the rigidity vanishes. Thus, by definition, one cannot gradually interpolate between two phases—they are only connected via the drastic event of a phase transition.

We shall here study properties of matter which is kept at such low temperatures that the thermal fluctuations can be neglected; the temperature is effectively zero. In a quantum system where the ground-state is separated from the first excited state by a finite energy gap, ΔE, this amounts to having $k_B T \ll \Delta E$, where k_B is Bolzmann's constant. Although there is a lot of current interest in gapless phases, we shall only consider those that are gapped.

Contrary to what you might think, not all matter solidifies even at the lowest temperatures. Many interesting phenomena such as superconductivity and superfluidity persist even down to zero temperature and this is due to quantum rather than thermal fluctuations. Phase transitions occur also at zero temperature, but are then driven by quantum rather than by thermal fluctuations, and are induced by changes in some external parameter. The transition from a superconductor to a normal metal by changing the magnetic field is a well know example of such a *quantum phase transition*. While the usual finite temperature phase transitions show up as abrupt changes in various thermodynamical quantities, quantum phase transitions are signaled by qualitative changes in the quantum mechanical ground-state wave function.

Since these quantum phases occur at zero temperature their properties are encoded in qualitative properties of the ground-state of the system. Thus, classifying and characterizing zero temperature states of matter is equivalent to doing this for ground-state wave functions with a very large number of degrees of freedom. More precisely, two systems at zero temperature are said to be in different phases when, in the thermodynamic, i.e. large volume, limit, there is no continuous way to transform one state (i.e., the ground-state wave function) into the other while remaining at zero temperature and not closing the energy gap. So when we say that one state cannot continuously be transformed into another we mean that this cannot be done in the thermodynamic limit while keeping the energy gap finite.

There are two complementary ways to precisely characterize (quantum) phases of matter. The first, which goes back to the work of Lev Landau in the 1930's, is based on symmetries [1]. The basic idea is that of *spontaneous symmetry breaking* which means that the ground-state has less symmetry than the microscopic Hamiltonian. An important concept is the order parameter, ψ_a, which transforms according to some representation of a symmetry group. The archetypal example is a ferromagnet where the order parameter is the magnetization, $\mathbf{M}$, which transforms as a vector under spatial rotations, and is non-zero below the Curie temperature. While the order parameter is essentially a classical concept, and the phase transitions studied by Landau are driven by thermal fluctuations, the Landau approach equally well applies to quantum systems. These can also be classified according to their pattern of spontaneous symmetry breaking, and the order parameter appears as the ground-state expectation value, $\psi_a = \langle \hat{\psi}_a \rangle$, where $\hat{\psi}_a(\mathbf{x},t)$ is an operator with appropriate symmetry properties that can be measured locally. Low-lying excitations around the symmetry broken ground-state corresponds to long wave length oscillations in $\psi_a(\mathbf{x},t)$ which are gapless if the broken symmetry is continuous—these are the famous Goldstone modes.

The second way phases can be distinguished, which is special to quantum systems, was developed in the last decades, and is drastically different from the Landau paradigm. Such phases differ by properties of the quantum entanglement of the ground-state wave functions, and they cannot directly be distinguished by local measurements. They are referred to as *topological phases of matter* or *topological quantum matter* and are the main topic of these notes.

Topology is the mathematical study of properties that are preserved under continuous transformations, and in this context it refers to properties of the ground-state wave function preserved under continuously changing external parameters. The distinctions between different topological phases of matter are somewhat intricate and require a systematic study using tools developed in the field of mathematical topology, motivating "topological" in the expression topological quantum matter.

There are two classes of topological phases with quite different properties: the *symmetry protected topological states* (SPT-states) and the *topologically ordered states* (TO-states). As already mentioned, two states are in different phases if one cannot continuously transform the ground-state wave function of one into that of the other. In many situations this turns out to be too restrictive and would not allow us to identify important topological phases. The reason is that there can be symmetries that *all* physically realizable perturbations respect. The natural question to ask is then what the possible phases are if one is restricted to systems with a certain symmetry—*symmetry protected phases.* That is, we widen the definition and say that two states which respect a specific symmetry S are in different SPT phases as long as any continuous transformations between them violate S.

A most striking property of the SPT states is that they support boundary states which can be used to classify them. The boundary state of a d-dimensional SPT state is described by a $d-1$ dimensional field theory which cannot describe a *bona fide* $d-1$ system which preserves the symmetry (we will use lower-case d to denote the number of spatial dimension, while upper-case D will denote the number of space-time dimensions, i.e., $D = d+1$). Typically, what happens is that the application of an external field makes the conserved charge, corresponding to the symmetry, flow from bulk to edge, thus preventing them from being individually consistent. As you will see, for some of these topological states this goes hand in hand with a quantization of certain transport coefficients, most notably the Hall conductance.

Symmetry protected phases have been known, at least as a theoretical possibility, since the work of Haldane on topological effects in $1d$ spin chains [2]. However, the field got a renaissance after the fairly recent both theoretical and experimental discovery of the time-reversal invariant topological insulators, see e.g. [3]. These states can be realized by non-interacting fermions and their discovery led to a systematic study of those states that continuously can be transformed into each other, restricting one self to *only* non-interacting systems. Soon, there was a complete classification of non-interacting fermionic systems with $U(1)$ symmetry and/or non-unitary symmetries in terms of non-interacting topological invariants [4, 5].

Since, in the real world, there are generally some interactions present, this classification would seem to apply only to very fine-tuned situations and be, at best, of marginal interest. Fortunately, however, for many (but not all) of the systems with non-interacting topological invariants there are characterizing properties, such as boundary states and quantized transport coefficients that do not dependent on interactions being absent. At a more theoretical level, it has also been shown that many of the non-interacting topological classes are characterized by various quantum anomalies, which are known to be insensitive to (at least weak) interactions [6].

The non-interacting classification has been essential for pinpointing these characteristics and is of great importance for the understanding of topological quantum matter in general. All SPT states that we will discuss in these notes can be realized with non-interacting fermions, but note that SPT phases appear in more general settings. For instance, all non trivial bosonic SPT states are interacting.

The meaning of TO has varied over time and there still no complete consensus concerning the nomenclature. However, some states are considered topologically ordered regardless of convention, namely the ones that support localized fractionalized bulk excitations with non-trivial topological interactions. The most famous examples are in $2d$ where these excitations are *anyonic* quasi-particles, *anyons*, which have a remarkable type of interaction. At first sight they do not seem to interact at all, at least not at long distances, but a closer look will reveal that there is a subtle form of interaction, referred to as anyonic, or fractional, statistics[1]: the state of the system will depend not only on the positions of the individual particles, but also on their history. Or more precisely, on how their world lines have *braided*.

In fact, one can classify the TO ground-states in terms of the properties of the quasi-particles they support without specifying the Hamiltonian. At first this may seem strange: If you only know the ground-state, any states could be made the low lying excited ones by judiciously picking the Hamiltonian. However, if you assume that the Hamiltonian is local, meaning a sum of local terms, $\hat{H} = \sum_{\{\mathbf{x}\}} \hat{h}_{\mathbf{x}}$ where each $\hat{h}_{\mathbf{x}}$ only has exponentially small support outside of a region close to $\mathbf{x}$, then there is a close connection between the properties of the ground-state, and those of the excitations, see e.g., [7, 8].

We discuss $2d$ states that support anyons but also touch upon analogous states in other dimensions. In $1d$, there are none, and in dimensions higher than two the states will support higher dimensional excitations (such as strings and membranes) that can realize higher dimensional analogs of anyons.

[1] The topological interaction between anyons is a generalization of the Berry phase of -1 acquired by the excange of two identical fermions. This minus sign is directly related to Fermi statistics and in the same way the topological interaction is related to a specific "exclusion statistics". This is the reason for the term "statistics" in anyonic statistics.

There are also phases that are topologically ordered in the sense that they are well defined without requiring the presence of a symmetry, but they have no excitations which interact topologically and their physical characteristics resemble that of SPTs. It is for these states that the conventions on whether they are topologically ordered or not vary. With an abuse of the concept they can be considered SPTs but the "symmetries" that protect them are not really symmetries, in the sense that they cannot be broken. An important example (that we will discuss) is the Majorana nano-wire [9] which can be considered protected by fermion parity conservation [10]. Fermion parity conservation is not really a symmetry since it cannot be broken, it is a property that always is present.

There are many examples of topological quantum matter that have been important for the development of the theoretical understanding of the field as well as being of great physical relevance. However, in these notes you will neither find an exhaustive list of such examples nor attempts to a complete classification. But rather focus on a few examples that we find particularly enlightening and relevant.

Since we only consider gapped local Hamiltonians we have characteristic time and length scales $\tau = \hbar/E_{gap}$ and the correlation length ξ. We will employ two different types of effective field theories. The first, which is valid at length and time scales at the order of ξ, τ and longer, will not capture the physics at microscopic scales (like the lattice spacing $a \sim \AA$ or the bandwidth $\Delta\varepsilon \sim eV$). Such theories are the analogs, for topological phases, of the Ginzburg-Landau theories used to describe the usual symmetry breaking non-topological phases. Examples are the Chern-Simons-Ginzburg-Landau theories for QH liquids, see e.g. [11], and the Ginzburg-Landau-Maxwell theories for superconductors[2]. These theories have information both about topological quantities, such as charges and statistics of quasiparticles, and of collective bosonic excitations such as plasmons or magnetorotons.

The second type of theories describe the physics on scales where non-topological gapped states would be very boring, namely at distances and times much larger than ξ and τ. On these scales everything is independent of any distance and the theories will be *topological field theories*, which do not describe any dynamics in the bulk, but do carry information about topological properties of the excitations, and also about excitations at the boundaries of the system. Typical examples we shall study are the Wen-Zee Chern-Simons theories for QH liquids [16, 17] and the BF-theories for superconductors and topological insulators [18, 19].

Finally, we will also study *effective response actions*. In a strict sense these are not effective theories, since they do not have any dynamical content, but encode the response of the system to external perturbations, typically an electromagnetic field. As you will see the effective response action for topological states can however be used to extract parts of the dynamic theory through a method called functional bosonization.

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[2] Most textbooks in condensed matter theory will cover the Ginzburg-Landau-Maxwell theory. For a modern text see e.g., [12]. Ref. [13], by S. Weinberg, one of the founders of effective field theory, gives a good presentation from the field theoretic point of view. There are also several excellent recent textbooks, as for instance [14, 15], on the general subject of these notes.

We have tried to make these notes reasonably self-contained, but of course we will often refer to other texts. The list of references is by no mean exhaustive; when there are good reviews we often cite these rather than the original papers.

1.2 The quantized Hall conductance

Figure 1.1 shows a schematic view of the experiment by von Klitzing, et. al. [20], where the quantum Hall effect was discovered, and also the original data. The quantum Hall effect will be discussed again later, but for now it suffices to know that the quantum Hall effect is observed in a two-dimensional electron gas, which is gated so that the chemical potential increases monotonically with the gate voltage V_g shown in figure 1.1. Von Klitzing's experiment was performed at a temperature of $1.5\,\mathrm{K}$ and a magnetic field of $18\,\mathrm{T}$. A constant current $I = 1\,\mu\mathrm{A}$ was driven through the system and the perpendicular voltage U_H was measured. In figure 1.1 you can see clear plateaux where the resistance is constant, and on a closer inspection these plateaux are located at integer multiples of the von Klitzing constant, R_K up to relative errors of the order $\lesssim 10^{-8}$.

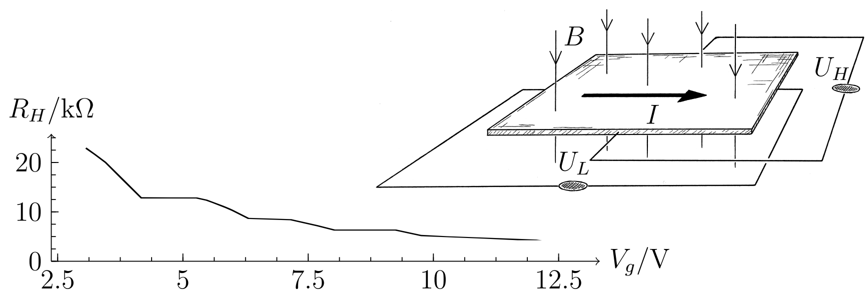

Fig. 1.1 Data from the original paper [20] with a set-up as is schematically depicted above: a two dimensional electron gas is subject to a magnetic field, B, and a constant current, I, is driven through the system. The chemical potential of the electron gas increases monotonically with V_g and there are clear plateaux with constant Hall resistance R_H. *Figure by S. Holst.*

Since then, similar experiments been performed a large number of times on different systems and in different parameter ranges including at room temperature [21]. The result is always the same, the Hall resistance is quantized at multiples or rational fractions of R_K. This is remarkable! Remember that we are dealing with macroscopic systems which depend on a practically infinite number of parameters. Even so, if you keep a constant current the voltage will be exactly the same in samples that can vary extensively. How can this be? In this section you will not only find the answer to this question and its related consequences, you will also become familiar with response actions, and topological field theories. These are important tools for analyzing topological states of matter, and they will be used extensively throughout these notes.

1.2.1 The Hall conductance as a Chern number

We now explain why the Hall conductance is quantized in gapped $2d$ system at temperatures $k_B T \ll \Delta E$. This is a most important fact: a quantized value cannot change continuously, so there has to be a phase transition between states with different Hall conductance—the quantized value is one of the phase-distinguishing characteristics mentioned in the introduction.

The argument given here is based on the work by Niu, Thouless, and Wu [22]. There are however many important earlier contributions that lay the ground work for the understanding, most notably [23–26].

By definition, the conductivity tensor gives the linear current density response to an electric field and in two spatial dimension it can be parametrized as,

$$j^i = \sigma^{ij} E_j = \sigma_H \varepsilon^{ij} E_j + \sigma_L E^i \ , \tag{1.1}$$

which defines the usual longitudinal conductivity σ_L, and the transverse, or Hall, conductivity σ_H. What is measured in an experiment, is however, not the conductivity, but the conductance for some macroscopic (or mesoscopic) sample. The conductance gives the current response to a voltage, or an electromotive force, $\mathscr{E}$, which is defined by,

$$\mathscr{E} = \int_\gamma d\mathbf{l} \cdot \mathbf{E} \ , \tag{1.2}$$

where the integral is along some curve γ. For an open curve, $\mathscr{E}$ is just the voltage difference between the endpoints, as in the Hall bar shown in figure 1.1, while in toroidal, cylindrical or Corbino geometry, the curve is closed, and the electromotive force should be understood as generated by a time dependent magnetic field through a hole, as in figure 1.2. Multiplying the definition of longitudinal and Hall conductances (1.1) with ε^{ij} and integrating along a curve γ gives,

$$I_\gamma = \int_\gamma d\mathbf{l} \cdot \mathbf{E} \, \sigma_H + \int_\gamma d\mathbf{l} \times \mathbf{E} \, \sigma_L \ , \tag{1.3}$$

wconductanceshere I_γ is the current passing through the curve γ. Assuming a homogeneous sample, so that the Hall conductivity is constant, this become

$$I_\gamma = \sigma_H \, \mathscr{E}_\gamma + \sigma_L \int_\gamma d\mathbf{l} \times \mathbf{E} \, , \tag{1.4}$$

which shows that in a pure $2d$ sample the Hall conductance actually equals the Hall conductivity which is a material property. In particular notice that no geometrical factors, that would be hard to measure with high precision, enter the relation.[3]

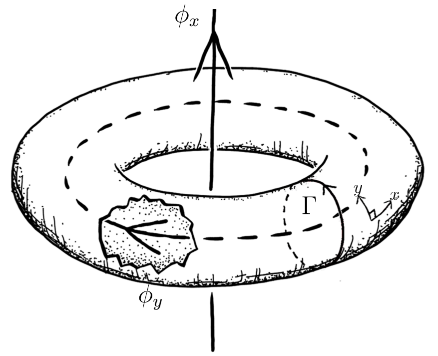

Fig. 1.2 A torus with flux ϕ_x/ϕ_x encircled by the two independent non-contractible loops and an example of a curve Γ which encircle the flux ϕ_y. *Figure by S. Holst.*

Now consider a $2d$ Hall bar which is large enough that the transport properties do not depend on the boundary conditions. We can then pick them to be periodic, which is the same as assuming that the space is a torus. Furthermore assume that the spectrum on the torus has a gap $\Delta E \gg k_B T$ above an N-dimensional subspace of degenerate states (up to splittings that vanish in the thermodynamic limit). We will also imagine having magnetic fluxes, $\phi_{x/y}$, through the torus as illustrated in figure 1.2. Our assertion is then that for a big system the Hall conductance will not depend on these fluxes, and will also equal that for a physical Hall bar.

Inserting an arbitrary flux, $\phi_{x/y}$, through any of the non-contractible loops on the torus (see figure 1.2) does not change the conductance, but it does change the Hamiltonian[4] . However, for the special case of inserting a flux quantum, $\phi_0 = hc/e$, the resulting Hamiltonian is identical to the one where there is no flux. Thus, the Hamiltonian depends on the parameters $\phi_{x/y}$ which are defined on a space where the points (ϕ_x, ϕ_y) and $(\phi_x + n_x\phi_0, \phi_y + n_y\phi_0)$ are identified. Put differently, the parameter space $T^2_\phi = \{(\phi_x, \phi_y)\}$ is a torus, which we will refer to as the flux-torus to distinguish it from the physical space, which, because of the periodic boundary conditions, also is a torus.

[3] This is not true in higher dimensions, and it is also not true for the longitudinal $2d$ conductance. Even for a pure sample does not equal the conductivity. For a rectangular Hall bar as in figure 1.1 the longitudinal conductance is $(W/L)\sigma$ where W and L are the widths and length of the bar respectively.

[4] In some references you will come across the notion of "twisted boundary conditions" this is equivalent to inserting flux through the holes of the torus.

The idea is now to consider maps from the parameter space into the space of ground-state wave functions. Such maps are characterized by an integer ch_1, called the first Chern number. The proper mathematical setting for this concept is the theory of fiber bundles, and more precisely, the degenerate ground-state wave functions form a complex vector bundle over the parameter space T^2_ϕ. For a brief introduction to Berry phases and Chern numbers we refer to section 1.8.1.

The basic result is that the Hall conductance σ_H is given by the formula,

$$\sigma_H = \sigma_0 \frac{ch_1}{N} \,, \tag{1.5}$$

where $\sigma_0 = e^2/h$ and N is the number of degenerate ground-states. We are now ready for the actual calculation.

Let us first pick the gauge potential as

$$\mathbf{A} = \tilde{\mathbf{A}} + \frac{\phi_x}{L_x}\hat{x} + \frac{\phi_y}{L_y}\hat{y} \,, \tag{1.6}$$

where the integral of $\tilde{A}_i$ along any of the non-contractible torus-loops is zero. Changing $\phi_{x/y} \to \phi_{x/y} + \phi_0$ would give back the same physical Hamiltonian but in a different gauge; so, $\phi_{x/y}$ is used not only label the fluxes through the holes, but also the gauge choice. Assume now that we have a monotonically increasing $\phi_y(t)$ such that in time τ a full unit of flux is inserted in the hole, i.e., $\phi_y(\tau) = \phi_y(0) + \phi_0$. According to Faraday's law, this generates an electromotive force,

$$\mathcal{E}_{\gamma_y} = \int_{\gamma_y} d\mathbf{l} \cdot \mathbf{E} = \frac{1}{c}\frac{\partial \phi_y}{\partial t} \,, \tag{1.7}$$

where γ_y is any of the non-contractible loops encircling the flux ϕ_y once. The conductance is defined in the limit of vanishing electric field so we should assume $\tau \to \infty$ and, since there is an energy gap to all excited states, the time dependence is thus given by the adiabatic theorem. We choose an orthonormal basis of the ground-state manifold, for each ϕ_x, at $\phi_y = \phi_y(0) = 0$,

$$\left\{ \left| (\phi_x, \phi_y); \alpha \right\rangle \right\}_{\alpha = 1,\ldots N} \Big|_{\phi_y = 0} \,, \tag{1.8}$$

which is taken as a smooth function of ϕ_x. Under the adiabatic time evolution $U(t) \equiv U(\phi_y(t))$ (recall that $\phi_y(t)$ is monotonic) this evolves into,

$$\left| (\phi_x, \phi_y); \alpha \right\rangle = U(\phi_y) \left| (\phi_x, 0); \alpha \right\rangle \,. \tag{1.9}$$

Now we are ready to calculate the current. With no loss of generality we shall take the curve γ to be a straight line in the y-direction, and get,

$$I_{\gamma_y} = \frac{1}{L_x}\int d^2x\,\hat{x}\cdot\mathbf{j}(\mathbf{x}) = \frac{1}{L_x}\int d^2x\,\psi^*(\mathbf{x},t)\frac{c\,\partial H(\mathbf{A})}{\partial A_x}\psi(\mathbf{x},t)$$
$$= \int d^2x\,\psi^*(\mathbf{x},t)\frac{c\,\partial H(\mathbf{A})}{\partial\phi_x}\psi(\mathbf{x},t) = c\left\langle\psi\left|\left(\partial_{\phi_x}H\right)\right|\psi\right\rangle\ , \quad (1.10)$$

where we used the definition of current density operator $\mathbf{j}(\mathbf{x}) = c\,\partial H(\mathbf{A})/\partial\mathbf{A}$, and our decomposition of the gauge potential, (1.6). We can, with out loss of generality, assume that we start out in the state $\left|(\phi_x,\phi_y);1\right\rangle$, and we then have the expression,

$$I_{\gamma_y}(\phi_y) = c\left\langle(\phi_x,\phi_y);1\left|\left(\partial_{\phi_x}H\right)\right|(\phi_x,\phi_y);1\right\rangle \quad (1.11)$$

which by, repeated use of Leibniz rule and using

$$H\left|(\phi_x,\phi_y);1\right\rangle = i\hbar\frac{\partial\phi_y}{\partial t}\partial_{\phi_y}\left|(\phi_x,\phi_y);1\right\rangle\ , \quad (1.12)$$

and $E_0 = \left\langle(\phi_x,\phi_y);1\left|H\right|(\phi_x,\phi_y);1\right\rangle$, can be rewritten as

$$I_{\gamma_y}(\phi_y) = c\partial_{\phi_x}E_0 + i\hbar c\frac{\partial\phi_y}{\partial t}\ . \quad (1.13)$$

When averaging over ϕ_y and τ, the first term vanishes since $E_0(\phi_x,\phi_y) = E_0(\phi_x + 2\pi,\phi_y)$ and we get

$$\bar{I}_{\gamma_y} = \frac{hc}{\phi_0}\frac{1}{\tau}\int_0^{\phi_0}\int_0^{\phi_0} d\phi_x d\phi_y \varepsilon^{ij}\partial_{\phi_i}\left\langle(\phi_x,\phi_y);1\right|\partial_{\phi_j}\left|(\phi_x,\phi_y);1\right\rangle\ . \quad (1.14)$$

The integrand equals a term in the trace of the Berry field strength corresponding to the state $\left|(\phi_x,\phi_y);1\right\rangle$, see (1.186) in section 1.8.1; so, if we average over the different states in the degenerate ground-state manifold we end up with

$$\bar{I}_{\gamma_y} = \frac{hc}{N\phi_0}\frac{1}{\tau}\int_0^{\phi_0}\int_0^{\phi_0} d\phi_x d\phi_y\frac{\mathrm{Tr}(\mathscr{F})}{2\pi} = \frac{1}{N}\frac{e}{\tau}ch_1\ , \quad (1.15)$$

where the first Chern number, ch_1, is defined in section 1.8.1. We now argue that without loss of generality we can do just that: Let us for simplicity assume that there are two ground-states, and compare their conductivity in some bounded region. (This can in principle can be measured by a local probe.) One possibility is that the conductivity is in fact the same in the two states, so that the conductance trivially equals the average of the conductances in the two states. The other possibility is that there is a region where the conductivities do differ, which means that there are *local* operators with different expectation values in the two states. Now think of weakly perturbing the Hamiltonian in some region with such terms. This will break the degeneracy, and result in a unique ground-state an the question of averaging is gone.

To get the conductance we are left with calculating the electromotive force. From Faraday's law, (1.7) we get,

$$\bar{\mathscr{E}}_y = \frac{1}{\phi_0 \tau c} \int_0^{\phi_0} \int_0^{\tau} d\phi_y dt \frac{\partial \phi_x}{\partial t} = \frac{\phi_0}{\tau c} \,, \tag{1.16}$$

and then finally,

$$\sigma_H = \frac{\bar{I}_x}{\bar{\mathscr{E}}_y} = \sigma_0 \frac{ch_1}{N} \,, \tag{1.17}$$

which concludes the proof of (1.5)—the quantization of the Hall conductance.

This formula allows for a conductance which is a fraction of the quantum of conductance σ_0, but only if the ground-state degeneracy cannot be broken by any local operator. This is in fact a characteristic property of topologically ordered states—the ground-state degeneracy is a topologically protected number. We will return to this point briefly in section 1.6, and you should note that this type of degeneracy is very different from what you are used to. Normally a degeneracy is either "accidental", that is dependent some fine tuning of parameters, or due to some symmetry. In both cases the degeneracy can be broken by adding local terms, which in the second case has to violate the symmetry.

As we just mentioned the states with a conductance of a fraction of σ_0, are topologically ordered. But what about the states which have a conductance of an integer times σ_0. Are they topologically ordered or symmetry protected?

The Hall conductance is a charge response and one could at least in principle imagine breaking the $U(1)$-charge conservation symmetry by proximity to a superconductor. The Hall-conductance would then no longer be well-defined and could no longer be used as a characteristic for a phase of matter. You might therefore think that the states with integer quantized Hall conductance are SPTs protected by $U(1)$-symmetry. And you would be right; One could, in principle, imagine an SPT with quantized Hall conductance. However, in all realistic situations the quantized charge Hall conductance goes hand in hand with a quantized *thermal* conductance: the heat current is proportional to a quantized constant, the temperature, and the temperature gradient. This quantized constant is harder to access experimentally than the charge Hall conductance, but it is by no means impossible, see e.g., [27]. However, it is the theoretical importance that is of main interest here; energy conservation is part of the definition of quantum matter (without it you could not define zero-temperature), so one cannot get rid of the thermal conductance in the same way as with the charge conductance. The states with integer quantized Hall conductance are therefore topologically ordered but of the kind, mentioned in the introduction, that are similar to SPTs.

1.2.2 The Chern-Simons response action

In this section we shall first encode the quantum Hall response, derived above, in an *effective response action*. From now on we put $c = 1$, and often $\hbar = 1$. We assume there is a conserved $U(1)$ current (typically the electric current) which we can couple to a gauge field A_μ. The effective action, $\Gamma[A_\mu]$, is the generating functional for connected n-point functions,

$$\Gamma[A_\mu] \equiv -i \log \mathscr{Z}[A_\mu] \equiv -i \log \left\langle GS \middle| \mathscr{T} e^{i\int dt H(A_\mu(t))} \middle| GS \right\rangle \ , \tag{1.18}$$

where $|GS\rangle$ is the ground-state and $\mathscr{T}$ denotes time-ordering. Taking derivatives of $\Gamma[A_\mu]$ gives time-ordered connected current n-point functions. Most importantly the current expectation value,

$$\frac{\delta}{c\,\delta A'_\mu(\mathbf{x},t)} \Gamma[A'_\mu] \Bigg|_{A'_\mu = A_\mu} \equiv j^\mu(\mathbf{x},t) = \left\langle GS \middle| \hat{j}^\mu(\mathbf{x},t) \middle| GS \right\rangle \ ; \tag{1.19}$$

$$\hat{j}^\mu(\mathbf{x},t) \equiv \mathscr{T} e^{i\int_t^\infty dt' H(A_\mu(t'))} \hat{j}^\mu(\mathbf{x}) \mathscr{T} e^{i\int_{-\infty}^t dt' H(A_\mu(t'))} \ , \tag{1.20}$$

where A_μ is a back-ground gauge potential.

Since we consider gapped systems, which by definition have have no mobile charge carriers, and thus do not conduct, we first remind ourselves of the effective action for usual insulators. In these materials, external electric and magnetic fields gives rise to dielectric and diamagnetic effects, such as a polarization charge, $\rho_{pol} = -\chi_e \nabla \cdot \mathbf{E}$. This *linear response* is captured by the effective action,

$$\Gamma_{med} = \int dt d^3x \left(\frac{\chi_e}{2} \mathbf{E}^2 - \frac{\chi_m}{2} \mathbf{B}^2 \right) , \tag{1.21}$$

where χ_e and χ_m are the electric and magnetic susceptibilities respectively. This is the expression with the lowest number of derivatives, which is quadratic in the fields (and thus gives linear response), and is invariant under rotations, reflections (parity), time reversal and gauge transformations. Thus it describes the response of a large class of isotropic materials in weak and slowly varying electromagnetic fields.

We now turn to the Hall response. The relation $j^i = \sigma_H \varepsilon^{ij} E_j$ can be written as

$$j^i = \sigma_H \varepsilon^{ij} \left(\partial_0 A_j - \partial_j A_0 \right) \ , \tag{1.22}$$

where 0 is the time index, the roman letters $i, j, \dots$ are the space indices. Eq (1.22) together with current conservation, $\partial_i j^i = \partial_0 j^0$ gives the *Streda formula*,

$$\partial_0 j^0 = \partial_i j^i = \sigma_H \varepsilon^{ij} \partial_i \left(\partial_0 A_j - \partial_j A_0 \right) = \sigma_H \partial_0 B \ ,$$

where B is the component of the magnetic field perpendicular to the $2d$ system. Assuming $B = 0$ at $t = -\infty$, this can be combined with (1.22) to give

$$j^\mu = \sigma_H \, \varepsilon^{\mu\nu\sigma} \partial_\nu A_\sigma \ , \tag{1.23}$$

and, by integration, the corresponding term,

$$\Gamma_H[A] = \frac{\sigma_H}{2} \int d^3x \, \varepsilon^{\mu\nu\sigma} A_\mu(x) \partial_\nu A_\sigma(x) \, , \tag{1.24}$$

in the effective response action.

As opposed to $\Gamma_{med.}$, the Hall term, Γ_H, violates both time-reversal and parity symmetry. Thus we can conclude that in a system where these symmetries are present we have zero Hall conductance. In the quantum Hall systems the symmetry is broken by a background magnetic field, but as you will see, a magnetic field is not necessary; other physical systems which violates the symmetry in other ways can also produce a non-zero Hall conductance.

Another point to notice is that the Hall term is not written only in terms of field strengths, so one might worry that it is not gauge invariant. Under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{1.25}$$

one gets the variation

$$\delta \int_V d^3x\, \varepsilon^{\mu\nu\sigma} A_\mu(x) \partial_\nu A_\sigma(x) = \int_{\partial V} dx^i E_i(x) \lambda(x) \;, \tag{1.26}$$

where ∂V is the boundary of the space-time volume V. The Hall term is thus gauge invariant only up to a boundary term. Since gauge invariance is a consequence of current conservation, this means that we do not have current conservation if the system of consideration has a boundary. The resolution to this quandary is that there is an extra piece in the effective action that only resides on the boundary and describes an edge current in the quantum Hall sample. Such edge currents are known to be present and it is important to find a formulation of the effective low energy theory that incorporates them in a natural way.

1.2.3 The topological field theory

The basic tool to find a formulation of the effective low energy theory will be that of *topological field theory*. We shall return to this concept several times later, but for now just look at the simplest example and see that it has the desired properties. We take the Lagrangian

$$\mathscr{L}(b;A,j) = -\frac{1}{4\pi} \varepsilon^{\mu\nu\sigma} b_\mu \partial_\nu b_\sigma - \frac{e}{2\pi} \varepsilon^{\mu\nu\sigma} b_\mu \partial_\nu A_\sigma - j_q^\mu b_\mu \;, \tag{1.27}$$

where b is an auxiliary gauge field[5], and j_q a quasi-particle current. The first term in (1.27) is called the Chern-Simons (CS) term, and this particular topological field theory is thus called CS theory. To understand the meaning of the field, b, we calculate the electric current j,

[5] Note that the conventions for this field differ. We use the notation from [11], while in the work by Wen [17] the field here denoted by b is denoted by a.

$$j^\mu = \frac{\delta \mathscr{L}}{\delta A_\mu} = -\frac{e}{2\pi}\varepsilon^{\mu\nu\sigma}\partial_\nu b_\sigma \ . \tag{1.28}$$

So, b is just a way to parametrize j. Note that j, which by definition is conserved, is invariant under the gauge transformation

$$b_\mu \to b_\mu + \partial_\mu \chi \,, \tag{1.29}$$

where χ is a scalar, since it is the field strength corresponding to the vector potential b. Since b is related to the conserved current, we shall refer to it as "hydrodynamic".

Why is this theory is referred to as topological? First you notice that it does not depend on the metric tensor. A normal kinetic term has the general covariant form $\sim g^{\mu\nu}D_\mu\phi D_\nu\phi$, and thus depends on the geometry of the space on which it is defined. If the action does not depend on the metric, correlation functions of operators cannot depend on the metric either, specifically they cannot depend on any distance or time. Furthermore, the equation of motion for the b field is,

$$\varepsilon^{\mu\nu\sigma}\partial_\nu b_\sigma = -e\varepsilon^{\mu\nu\sigma}\partial_\nu A_\sigma - 2\pi j_q^\mu \ , \tag{1.30}$$

that is *the field strength is completely determined by the external sources*. This means that, as opposed to usual Maxwell electrodynamics, there are no propagating photons—the equations of motion are just constraints. For instance, the zeroth component of (1.30) is $2\pi\rho = \varepsilon^{ij}\partial_i b_j \equiv B^{(b)}$, which relates $B^{(b)}$, to the charge density, $\rho = j^0$, of the external sources. The analysis just given is, only for a system on an infinite plane, the case of boundaries will be discussed below, in section 1.2.4.

Since the Lagrangian, (1.27), is quadratic in b we can integrate it out to get an effective action for A only. We can use the following path integral formula,

$$e^{i\Gamma[A,j]} = \int \mathscr{D}[\mathbf{b}]e^{i\int d^3r\,\mathscr{L}(b;A,j)} \ , \tag{1.31}$$

to get the response action. Performing the integral we get,

$$\begin{aligned}\Gamma[A,j] = \int d^3x \left[\frac{\sigma_H}{2}\varepsilon^{\mu\nu\sigma}A_\mu(x)\partial_\nu A_\sigma(x) + e\, j_q^\mu(x)A_\mu(x)\right] \\ + \int d^3x d^3y\, j_q^\mu(x)\left(\frac{\pi}{d}\right)_{\mu\nu}(x-y)\, j_q^\nu(y) \ ,\end{aligned} \tag{1.32}$$

where you should recall that $\sigma_H = e^2/2\pi$, and where $(1/d)_{\mu\nu}(x-y)$ is the inverse of the Chern-Simons operator kernel $\varepsilon^{\mu\nu\sigma}\partial_\sigma$. The first term in this expression is just the Chern-Simons response term, (1.24), derived earlier, while the last term is a *topological interaction* between the particles described by the source j_q. The last term provides the minus sign that the wave function acquires when two identical fermions are exchanged. A simple way to understand this phase is to recall that the equation of motion (1.30) associates charge with flux and that the resulting charge-flux composites will pick up an Aharanov-Bohm like phase when encircling each other. (The story is a little more subtle and we will come back to in the last section on the fractional quantum Hall effect.) We again stress that the above result is only correct on an infinite plane, since it does not conserve current at the edge.

1.2.3.1 Functional bosonization

You might find the above discussion somewhat unsatisfactory since the topological field theory in the previous section was merely postulated. Here we amend this by describing a rather general method to actually *derive* a topological field theory given an effective response action [28]. After a general exposition of the method we then specialize to Hall response.

The starting point is the path-integral formula for the partition function,

$$\mathscr{Z}[A_\mu] = \int \mathscr{D}[\bar{\psi},\psi] e^{iS[\bar{\psi},\psi,A]} \,, \tag{1.33}$$

where the action S describes the system of interest. We will not do the integral explicitly, but make use of the fact that for a gapped system at zero temperature the functional $\mathscr{Z}[A_\mu]$ will be local.

Because of current conservation, the effective response action, and therefore also $\mathscr{Z}$, needs to be gauge invariant meaning that

$$\mathscr{Z}[A_\mu + a_\mu] = Z[A_\mu] \,, \tag{1.34}$$

for any a being a pure gauge, i.e. satisfying,

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\mu a_\nu = 0 \,. \tag{1.35}$$

Thus one can express $\mathscr{Z}$ as

$$\mathscr{Z}[A] = \int \mathscr{D}[a]\, \mathscr{Z}[A+a] \prod_{\mu\nu\cdots} \varepsilon^{\mu\nu\lambda\ldots\alpha\beta} \delta[f_{\alpha\beta}(a(x))] \,, \tag{1.36}$$

where the delta functionals under the product sign enforce the zero field-strenght constraint. Here, x is a point in $D = d+1$ dimensional space-time, and $\varepsilon^{\mu\nu\lambda\ldots\alpha\beta}$ is the D-dimensional totally anti-symmetric Levi-Civita symbol. Introducing an auxiliary tensor field $b_{\mu_1\mu_1\ldots\mu_{D-2}}$ to express the delta functional as a functional Fourier integral, we get,

$$\mathscr{Z}[A] = \int \mathscr{D}[a]\mathscr{D}[b]\, \mathscr{Z}[A+a] e^{i\frac{1}{2}\int d^D x \varepsilon^{\mu\nu\lambda\ldots\alpha\beta} b_{\mu\nu\lambda\ldots} f_{\alpha\beta}(a)} \,, \tag{1.37}$$

and by the shift $a \to a - A$, finally,

$$\mathscr{Z}[A] = \int \mathscr{D}[a]\mathscr{D}[b]\mathscr{Z}[a]e^{i\frac{1}{2}\int d^Dx\,\varepsilon^{\mu\nu\lambda\ldots\alpha\beta}b_{\mu\nu\lambda\ldots}[f_{\alpha\beta}(a)-F_{\alpha\beta}(A)]}$$
$$\equiv \int \mathscr{D}[a]\mathscr{D}[b]e^{i\int d^Dx\mathscr{L}}\ , \quad (1.38)$$

where the last equality defines the Lagrangian $\mathscr{L}$. Note that this action, by construction, is invariant under gauge transformations of the electromagnetic potential A_μ, since the electrical current is conserved in the model of consideration. Below you will see that this implies the existence of edge modes.

Given this one can calculate the expectation value of the $U(1)$ current as

$$\langle j^\mu(x)\rangle = i\frac{\delta \ln \mathscr{Z}[A]}{\delta A_\mu(x)} = \left\langle \varepsilon^{\mu\nu\lambda\rho\cdots}\partial_\nu b_{\lambda\rho\ldots}(x)\right\rangle\ , \quad (1.39)$$

and similarly for higher order correlation functions. Note that by construction the current is conserved. To appreciate the meaning of the field $b_{\mu\nu\lambda\ldots}$, let us look at the simplest special cases. For $D=2$, b is a scalar, and the above relation reads,

$$\langle j^\mu(x)\rangle = \langle \varepsilon^{\mu\nu}\partial_\nu b(x)\rangle\ , \quad (1.40)$$

which you might recognize if you are familiar with the method of bozonization in 1+1 dimension. This case is special, in the sense that it holds even if the average is removed, that is it holds as an *operator* identity. A concise account of the fascinating physics and mathematics of $(1+1)D$ systems can be found in the books [14, 29].

For $D=3$, b is vector field and

$$\langle j^\mu(x)\rangle = \langle \varepsilon^{\mu\nu\sigma}\partial_\nu b_\sigma(x)\rangle\ . \quad (1.41)$$

Up to a normalization, which we will discuss below this is the same as the previously derived relation for the electric current, (1.28).

Clearly the expression for $\mathscr{Z}[A_\mu]$ derived above, (1.38), is useful only if we can, at least approximately, evaluate the fermionic functional integral to get $Z[a]$. In 1+1 dimensions this can sometimes be done exactly. In higher dimensions this is not possible. However one can find an approximation by assuming there is a gap and making a derivative expansion.

In the case of $2d$ systems with quantum Hall response, we already know one piece in $Z[a]$ that will for sure be present namely the Hall term, (1.24). Combining this with the just derived expression for $\mathscr{Z}[A_\mu]$ (1.38), gives the effective Lagrangian,

$$\mathscr{L} = -\frac{1}{2\pi}\varepsilon^{\mu\nu\sigma}b_\mu\partial_\nu(a_\sigma - A_\sigma) + \frac{\sigma_H}{2}\varepsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma\,, \quad (1.42)$$

where we renormalized the field b so that it, up to the factor $(-e)$, is identical to the previous expression for the electromagnetic current, (1.28).

Above we used a seemingly arbitrary argument to fix the the normalization of the field b, and you should worry about this since a different convention would give a different value for σ_H when b is integrated over to obtain the effective response action. To understand this point, we must look closer at the first term in the above Lagrangian,(1.42) which is the 2d incarnation of the topological *BF theory* which is defined in any dimension as,

$$\mathscr{L}_{BF} = -\frac{1}{2}\varepsilon^{\mu\nu\lambda\ldots\alpha\beta} b_{\mu\nu\lambda\ldots} f_{\alpha\beta}(a)\,. \tag{1.43}$$

In section 1.6.2 we shall briefly discuss the $3d$ case in the connection with fluctuating superconductors. There is a rich mathematical literature on BF theory, [30, 31], but here we shall only cover material of direct relevance for physics. One such point is the question of normalization brought up above. With the chosen normalization the ground-state is unique, as is required for a number of filled Landau levels. A derivation of this result is given in section 1.8.2.

1.2.4 The bulk-boundary correspondence

We now show how the Chern-Simons topological field theory (1.27) in a natural way incorporates the presence of edge excitations. For a more thorough discussion you should consult the paper by Stone [32] and the review by Wen [33]. We specify the action by integrating the CS Lagrangian, (1.27), over a bounded and simply connected region D, and for simplicity we will put $\sigma_H = \sigma_0$ throughout this section,

$$S[b;A] = \int_D d^3x\, \mathscr{L}(b,A)\,. \tag{1.44}$$

This action is not gauge invariant since we get a non-zero variation at the boundary ∂D. What this means is that the pure gauge mode, $\partial_\mu\chi$ which in the bulk has no physical meaning, (and would be removed by gauge fixing) will at the edge manifest itself as a physical degree of freedom. To see this explicitly we substitute $b_\mu = \partial_\mu\chi$ into the Lagrangian to get,

$$S[b,\chi;A] = -\frac{1}{4\pi}\int_D d^3x\, \varepsilon^{\mu\nu\sigma} b_\mu \partial_\nu b_\sigma + \int_{\partial D} dt dx\, \chi\, \partial_x(\partial_t - v\partial_x)\chi(x,t)\;, \tag{1.45}$$

where for simplicity we neglected the external field A (which is easily reintroduced), and where the field $\chi(x,t)$ has support only on the boundary ∂D parametrized by the coordinate x. We also added an extra term $\sim \chi\,\partial_x^2\chi$ that does not follow from the Chern-Simons aciton (1.44), but which will be present if there is an electrostatic confining potential [33], which is needed e.g. in the quantum Hall effect to define the quantum Hall droplet. The meaning of this term is clear from the equation of motion for the χ-field,

$$(\partial_t - v\partial_x)\chi(x,t) = 0\,, \tag{1.46}$$

which shows that v is the velocity of a gapless edge-mode propagating in one direction. The physical origin of this velocity is obvious: it is the $\mathbf{E} \times \mathbf{B}$ drift velocity of the electrons in the external magnetic field and the confining electric field at the boundary. If we reintroduce the electromagnetic field and study the current conservation at the boundary, we will see that the non-conservation of the bulk current, which follows from the non gauge-invariant part of the Chern-Simons action, (1.26), is compensated by a corresponding non-conservation of the boundary current; so the total charge is conserved [32].

The mathematics related to this result is quite interesting. The boundary theory should after all just be a model for electrons moving in one direction along a line, and as such we would expect the theory just to be that of a Fermi gas, or if interactions are present, a Luttinger liquid. In both cases we would expect the boundary charge to be conserved. What is special here is that the mode is *chiral*, i.e. it only propagates in one direction. From the theory of Luttinger liquids, we learn that in the presence of an electric field, the right and left moving currents are not separately conserved, but only their sum, which is the electromagnetic current. The difference, which defines the *axial current*, which in Dirac notation reads $j^A_\mu = \bar{\psi}\gamma_3\gamma_\mu\psi$, is not conserved because of the *axial anomaly*. The subject of anomalies in quantum field theory is fascinating, but will not be discussed in these lectures.

1.3 Physical systems with quantized Hall conductance

In this section we move from the general discussion to real physical systems that have a quantized Hall response. We begin with the integer quantum Hall effect (IQHE), which started the whole field of topological states of matter. Given the previous general analysis, we can use a simple symmetry argument to explain it.

We then turn to the systems that have been game changers for the last decade—the various kinds of topological band insulators. Recall that one of the early successes of quantum mechanics was the division of crystalline materials into conductors, semiconductors and insulators depending on whether or not the Fermi level is inside a band gap (there is no sharp distinction between insulators and semiconductors; only a conventional classification depending on the size of the gap). The insulators seem to be the most boring states, and it was an important discovery that they can belong to different topological classes which differ in their quantum Hall response.

In an intermediate step, we study a system with both a magnetic field, and a weak periodic potential. This is instructive not only for providing a more realistic model for the quantum Hall effect, but also for showing how to calculate a topological invariant for a clean (non-interacting) band insulator. We then turn to the first example a system with a quantized Hall effect without any magnetic field, the Chern Hall insulator, and stress the importance of breaking time-reversal invariance.

1.3.1 The integer quantum Hall effect

The integer quantum Hall effect is observed when a very clean two dimensional electron gas is cooled and subjected to a strong perpendicular magnetic field.

If we neglect electron-electron interactions, this is the famous Landau problem and we know that the energy is quantized as $E_n = n\hbar\omega_c$ with the cyclotron frequency $\omega_c = eB/m$. Each of these Landau levels (LLs) has a macroscopic degeneracy such that there is one quantum state in the area $2\pi\ell_B^2 \equiv 2\pi\hbar/eB$ which corresponds to a unit flux, $\phi_0 = h/e = 2\pi/e$.

In section 1.2.1 you learnt that for a gapped state the Hall conductance is always quantized in integer multiples of the quantum of conductance divided by the degeneracy of the ground-state. For a number of filled Landau levels, the ground-state is non-degenerate, and one can obtain the Hall conductance by a simple symmetry argument: Let us start from the idealized case of non-interacting electrons moving in the xy-plane, and no impurities. Having p completely filled LLs corresponds to a charge density $\rho = -ne$, where $n = p/(2\pi\ell_B^2)$ is the electron number density. From this we get

$$B = \frac{\rho}{p}\frac{2\pi\hbar}{e^2} = \frac{1}{p\,\sigma_0}\rho \ , \tag{1.47}$$

where $\mathbf{B} = B\hat{z}$.

Now assume that in our frame we have an electric field $\mathbf{E} = 0$ and vanishing current density $\mathbf{j} = 0$. Then consider a frame moving with velocity $-\mathbf{v}$, $v \ll c$, relative to us. In this frame $\mathbf{B}$ is unchanged, but $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$, so

$$\mathbf{j} = \rho\,\mathbf{v} = p\sigma_0\,\hat{z}\times\mathbf{E} \ , \tag{1.48}$$

from which follows that the Hall conductivity is $\sigma_H = p\sigma_0$, and since our system is invariant under Galilean transformation, this result holds in any inertial frame. We now return to a realistic system with electron-electron interactions, and impurities. Since we have already shown that the conductance is quantized as long as the gap remains we know that the conductance must stay the same as these potentials are turned on under the assumption that the gap does not close.

1.3.2 The Hall conductance in a periodic potential

We shall now present a special case of a problem originally treated in a very influential paper by Thouless et. al. [24]. Recall that in a constant magnetic field, the (magnetic) translation operators, T_1 and T_2, which commute with the Hamiltonian, do not commute among themselves. This follows because $T_1^{-1}T_2^{-1}T_1T_1$ does not equal identity, since it amounts to a closed path that encloses flux and thus by the Aharonov-Bohm argument gives a phase to the wave function. From this we learn that if one picks a *flux lattice*, which is defined such that there is exactly one flux-quantum through a unit cell, the lattice translation operators will all commute and can be simultaneously diagonalized.

We shall consider the case where the Bravais lattice of the periodic potential is such that the unit cell of the flux lattice contains an integer number of unit cells of the potential[6]. In this case we can still diagonalize the magnetic translations, and then invoke Bloch's theorem to express the wave functions as,

$$\psi_{\mathbf{k}n}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} u_{\mathbf{k}n}(\mathbf{x}) \ , \tag{1.49}$$

where n is a band index (here the LL index) and $\mathbf{k}$ the crystal, or quasi, momentum that lives in the "magnetic Brillouin zone", $|k_i| \leq \pi/\ell_B = \pi\sqrt{eB}$.[7] The Bloch functions $u_{\mathbf{k}n}(\mathbf{x})$ are eigenfunctions of the Bloch Hamiltonian,

$$H_{Bl} = \frac{\hbar^2}{2m}(-i\nabla + e\mathbf{A} + \mathbf{k})^2 + V_{lat} \, . \tag{1.50}$$

We will assume that the periodic potential is weak enough that the cyclotron gap persist, and that the lowest N bands are completely filled. For each $\mathbf{k}$ in the Brillouin-zone (BZ) there are N wave functions

$$\left\{ e^{i\mathbf{k}\cdot\mathbf{x}} u_{\mathbf{k}n}(\mathbf{x}) \right\}_{n=1,\dots,N} \ , \tag{1.51}$$

so associated to each $\mathbf{k} \in BZ$ there is an N dimensional sub-Hilbert-space, $h(\mathbf{k})$, of the full single particle Hilbert space. The set $\{\mathbf{k}, h(\mathbf{k})\}_{\mathbf{k}}$ where $\mathbf{k} \in BZ$, is thus a fiber bundle over the Brillouin-zone, see section 1.8.1. By definition the Berry connection on this fiber bundle is

$$\mathscr{A}^{nm}_{k_i}(\mathbf{k}) = -i \langle \psi_{\mathbf{k}n}(\mathbf{x}) | \partial_{k_i} | \psi_{\mathbf{k}m}(\mathbf{x}) \rangle \equiv -i \int d^2x \, \psi^*_{\mathbf{k}n}(\mathbf{x}) \partial_{k_i} \psi_{\mathbf{k}m}(\mathbf{x}) \, .$$

This can also be written as the anti-commutator of the creation and annihilation operators,

$$\mathscr{A}^{nm}_i = -i \left\{ a_{\mathbf{k}n}, \partial_{k_i} a^\dagger_{\mathbf{k}m} \right\} \tag{1.52}$$

where $a^\dagger_{\mathbf{k}n}$ and $a_{\mathbf{k}n}$ are the Fourier components of the electron creation and annihilation operators $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$ satisfying $\left\{ \psi^\dagger(\mathbf{x}), \psi(\mathbf{x}') \right\} = \delta^2(\mathbf{x} - \mathbf{x}')$. The corresponding Berry field-strength, which we denote by $\mathscr{B}$ to distinguish it from the previously defined flux-torus field strength, becomes $\mathscr{F}$)

$$\mathscr{B}^{nm}_{k_i,k_j} = \partial_{k_i} \mathscr{A}^{nm}_{k_j} - \partial_{k_i} \mathscr{A}^{nm}_{k_j} + i \mathscr{A}^{np}_{k_i} \mathscr{A}^{pm}_{k_j} - i \mathscr{A}^{np}_{k_i} \mathscr{A}^{pm}_{k_j} \ , \tag{1.53}$$

[6] Ref. [24] treated the case where the ratio between the areas of the unit cells in the flux lattice and the Bravais lattis of the potential is a rational number q/p. In case one has to consider a larger unit cell, and each filled band will in general have a larger Hall conductance.

[7] In a translationally invariant system, the shape of this zone is arbitrary, but the area is fixed to support n units of magnetic flux. In our case the shape has to be taken as to be commensurate with the Bravais lattice of the potential.

where the repeated index p should be summed over. Since the Brillouin-zone is two-dimensional, the field strength has only one independent component

$$\mathscr{B} \equiv \varepsilon^{ij} \mathscr{B}^{nm}_{k_i,k_j} \ . \tag{1.54}$$

From now on we will suppress the upper indices m,n etc. and multiplications of $\mathscr{B}$ or $\mathscr{A}$ has the meaning of matrix multiplication. In this short hand notation we have

$$\mathscr{B} = \varepsilon^{ij} \left(\partial_{k_i} \mathscr{A}_{k_j} + i \mathscr{A}_{k_i} \mathscr{A}_{k_j} \right) \ , \tag{1.55}$$

for the the Berry "magnetic" field in the Brillouin-zone.

Although this *Brillouin zone bundle* is conceptually quite different from the flux bundle, with Berry curvature $\mathscr{F}$, introduced in section 1.2.1, they turn out to be closely related in the present case where electron-electron interactions and random impurities are neglected. To see this, first recall that the non-interacting many body ground-state is given by

$$|GS\rangle = \prod_{n=1}^{N} \prod_{\mathbf{k} \in BZ} a^{\dagger}_{\mathbf{k}n} |0\rangle \ . \tag{1.56}$$

Secondly, from the Bloch Hamiltonian, H_{Bl}, we can, by taking a vector potential describing fluxes through the holes in the torus, infer that the Bloch functions at finite flux are related to those at zero flux by,

$$u^{\phi_x,\phi_y}_{\mathbf{k}n}(\mathbf{x}) = u^{0,0}_{\mathbf{k}'n}(\mathbf{x}) \qquad ; \qquad \mathbf{k}' = \left(k_x + \frac{2\pi}{L_x} \frac{\phi_x}{\phi_0}, k_y + \frac{2\pi}{L_y} \frac{\phi_y}{\phi_0} \right) \ , \tag{1.57}$$

where ϕ_x and ϕ_y denote the fluxes encircled by the two independent non-contractible loops on the torus, see figure 1.2 . This means that a derivative with respect to a the flux ϕ_i can be turned into a derivative with respect to the crystal momentum k_i. We can also define the Chern number for the flux-torus fiber bundle, defined by the N first LLs,

$$|GS,\phi\rangle = \prod_{n=1}^{N} \prod_{\mathbf{k} \in BZ} a^{\dagger}_{\mathbf{k}n\phi} |0\rangle \qquad ; \qquad a^{\dagger}_{\mathbf{k}n} = \int d^2x\, \psi^{\dagger}(\mathbf{x}) \psi^{\phi_x,\phi_y}_{\mathbf{k}n}(\mathbf{x}) \ , \tag{1.58}$$

i.e., with Berry connection

$$\mathscr{A}_{\phi_i} = \langle GS,\phi | \partial_{\phi_i} | GS,\phi \rangle \ . \tag{1.59}$$

In section 1.2.1 we showed that the Chern number of this connection is proportional to the Hall conductance, and in section 1.8.1.4 we show that the first Chern number for the flux-torus fiber bundle, and the Brillouin zone bundle are equal. Combining these results yields the formula

$$\sigma_H = \frac{\sigma_0}{2\pi}\int d^2k\,\mathrm{Tr}\,\mathscr{B}\,, \tag{1.60}$$

for the Hall conductance. Thus, in this case we can express the Hall conductance in terms of the first Chern number of the BZ bundle. This result was originally obtained in [24] by using linear response and properties of the Bloch wave functions.

The above calculation using the Brillouin bundle relies on translational invariance, and the absence of interactions and is thus much less general than the result derived in section 1.2.1. When applicable the formula derived here is however much simpler to handle, and it has been essential in developing the classification of non-interacting topological matter—its theoretical importance should not be underestimated.

1.3.3 The Chern insulator

The formula (1.60) for the Hall conductance opens the possibility of having a topological phase in a crystalline system even in the absence of a magnetic field. Importantly, it demonstrates that *topological band theory* can be used to determine the actual values of the topological invariants.

In an important paper from 1988, Haldane showed that one can have a quantum Hall effect without any net magnetic field [34]. He constructed an effective model of electrons hopping on a hexagonal lattice penetrated by a staggered magnetic field that is on average zero. The model did, however, break time-reversal invariance and is today referred to as a *Chern insulator* that exhibits a *quantized anomalous Hall effect*. A model that is slightly simpler than the one used by Haldane is free electrons hopping on a square lattice, with a π-flux on each elementary plaquette [35]. Since the main theme of these notes are continuum field theory descriptions, we will not give the position space lattice Hamiltonian, which you can find in the original work [35]. For the present purpose, it suffices to say that *the Chern insulator* can be modelled by the following two-band momentum space Hamiltonian,

$$H_C = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} h^{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\beta}\,, \tag{1.61}$$

with

$$h^{\alpha\beta}(\mathbf{k}) = d_a(\mathbf{k})\sigma^a_{\alpha\beta} \qquad ; \qquad \mathbf{d} = (\sin k_x, \sin k_y, M + \cos k_x + \cos k_y)\,, \tag{1.62}$$

where both energy and M are measured in units of some hopping strength, t.

To calculate the Berry flux, we note that $h^{\alpha\beta}(\mathbf{k})$ is nothing but the Hamiltonian for a spin-half particle moving in a magnetic field $\mathbf{d}$. The spectrum, given by the Zeeman energy, is thus $\pm|\mathbf{d}|$, and the eigenfunctions satisfy,

$$\hat{d}\cdot\sigma\,|E_k;\pm\rangle = \pm\,|E_k;\pm\rangle\,. \tag{1.63}$$

Assuming that there is no gap closing, i.e., $|\mathbf{d}| > 0$, and taking the Fermi energy to be zero, the Berry field strength can be shown to be

$$\mathscr{B}^{\pm}(\mathbf{k}) = \varepsilon_{ij}\partial_i \mathscr{A}^{\pm}_j = \mp\frac{1}{2}\varepsilon_{ij}\hat{d}\cdot\partial_i\hat{d}\times\partial_j\hat{d}\,, \tag{1.64}$$

where the lower sign corresponds to the filled band and we used the short hand notation $\partial_{i/j} \equiv \partial_{k_{i/j}}$. The most direct way to show the above formula, although a bit tedious, is to first find $|E_k;-\rangle$ and then just calculate. An alternative derivation that does not require the explicit wave functions is given in section 1.8.3.

The expression on the right hand side of the above expression is the Jacobian of the function $\hat{d}(\mathbf{k})$, and we define the integer valued Pontryagin index by

$$\mathscr{Q} = \frac{1}{8\pi}\int d^2k\,\varepsilon_{ij}\hat{d}\cdot\partial_i\hat{d}\times\partial_j\hat{d}\,, \tag{1.65}$$

which measures how many times the surface of the unit sphere on which $\hat{d}$ is defined, is covered by the map from the compact manifold where $\mathbf{k}$ is defined.

It remains to determine the value of $n = \mathscr{Q}$. For large $|M|$, where the hopping can be neglected, the eigenfunctions $|E_k\rangle$ become $\mathbf{k}$-independent and the Pontryagin density is identically zero. ($M \gg 1$ is the atomic limit where the wave functions are sharply localized at the lattice sites.) $\mathscr{Q}$ is a topological quantity, so it can only change when the gap in the Fermi spectrum closes and $\hat{d}$ no longer is a smooth function of $\mathbf{k}$. From the expression $E_k = -|\mathbf{d}(\mathbf{k})|$, one realizes that the gap closes for $M = -2$ (at $\mathbf{k} = 0$), for $M = 2$ (at $\mathbf{k} = (\pi,\pi)$) and for $M = 0$ (at $\mathbf{k} = (0,\pi)$ and $\mathbf{k} = (\pi,0)$).

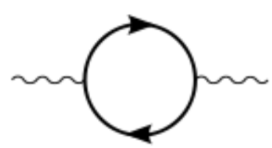

Fig. 1.3 Diagram giving rise to the CS action as the lowest term in a derivative expansion.

Let us now analyze what happens when M increases from a large negative value towards -2. Putting $M = -2+m$, and linearizing the Hamiltonian one gets,

$$H_{lin} = k_x\sigma_1 + k_y\sigma_2 + m\sigma_3\,, \tag{1.66}$$

which we recognize as the Hamiltonian for a $D = 2+1$ Dirac particle. Since the topological nature of a phase is a low-energy property, one should be able to capture the change in phase by analyzing the continuum theory in the vicinity of $m = 0$. We now show how to do this.

The continuum D=2+1 Dirac theory is defined by the Lagrangian,

$$\mathscr{L}_D = \bar{\psi}\left(\gamma^\mu (i\partial_\mu - eA_\mu) - m\right)\psi \tag{1.67}$$

with $\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1)$. We calculate the electromagnetic response by integrating out the fermions, which, to lowest order in A_μ, amounts to calculating the loop diagram in Fig. 1.3. This was originally done by Redlich [36] with the result,

$$\Gamma_D[A] = \frac{m}{|m|}\frac{e^2}{8\pi}\int d^3x\, \varepsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma\,. \tag{1.68}$$

Note that a Dirac mass term in $2d$ breaks the parity symmetry, so it is not surprising that a Chern-Simons term appears in the effective action. The term persists even in the limit $m \to 0$, where the classical Lagrangian respects parity, and is therefore often referred to a parity anomaly.[8] In section 1.8.4 we give an alternative derivation of this result by calculating the response to a constant magnetic field. Just as the Schrödinger case, the eigenvalues of the Dirac equation falls into Landau levels, $E_n = \pm\sqrt{neB + m^2}$, for $n > 0$, and the contribution from these states to σ_H cancel. Only the lowest Landau level, with $n = 0$ and energy $E_0 = m$, contributes. This derivation stresses that even though anomalies seem to be related to the short distance behaviour of the theory, they should be considered as a low energy effect.

Note that the coefficient in the response action derived from the Dirac Lagrangian,(1.68), translates into a Hall conductivity $\sigma_H = \pm\sigma_0/2$ which is half of the one calculated above. This is surprising since we have argued that the Hall conductivity, for topological reasons, must be an integer times σ_0. The solution to this apparent contradiction, is that it is not possible to consistently formulate the Dirac equation on a lattice without "doubling" the number of low-energy fermions. This result, first obtained in the context of high energy physics by [37], basically says that the low-energy physics of fermions in a band of finite width, cannot be faithfully represented by a single Dirac field[9]. We now return to the two-band model (1.61). Recall that we put $M = -2 + m$, and we want to know what happens as m is tuned from a small negative value to a small positive value. Doing this changes the spectrum only in the close vicinity of $\mathbf{k} = 0$, so the *change* in σ_H should be faithfully represented by the linearized Dirac theory. From the response action derived from the Dirac Lagrangian (1.68) we get the change $\Delta\sigma_H = \frac{1}{2}(1 - (-1))\sigma_0 = \sigma_0$. A similar analysis can be made for the other gap closing points. The result is that σ_H, in units of σ_0 changes as $0 \to 1 \to -1 \to 0$ as M is tuned from $-\infty$ to ∞. It should now also be clear, that the effective topological theory for the Chern insulator is identical to that for the IQHE given by (1.42). Note that at $M = 0$ the gap closes at two points, so to model this transition we need two Dirac fields, and thus the change of two units in σ_H.

In 2013, a quantized Hall effect was observed in thin films of Cr-doped $(\mathrm{Bi,Sb})_2\mathrm{Te}_3$ at zero magnetic field, thus providing the first experimental detection of a Chern insulator [38].

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[8] As explained in [36] a gauge invariant regularization of the ultraviolet divergence (e.g. using the Pauli-Villars method) gives rise to the anomaly term, but does not fix the sign.

[9] See section 16.3.3 in [14] for more details.

1.4 Generalizing to other dimensions

1.4.1 The $1d$ *case*

Until now we discussed the $2d$ systems with $U(1)$ symmetry and showed that in the topological scaling limit, their response action is the Hall term

$$\Gamma[A_\mu] = \frac{\sigma_H}{2} \int d^3x\, \varepsilon^{\mu\nu\sigma} A_\mu F_{\nu\sigma} \,, \tag{1.69}$$

which encodes the Hall response. We again emphasize that this expression is independent of the metric.

In $2d$ we knew from the start that we where looking for, a Hall response. Now assume that we had not known that, but would anyway have asked the question: Is there a possible topological response? A strategy in this hypothetical case would have been to write down all possible response actions that are gauge invariant and independent of the metric (and thus of any length or time scale). On a technical level the absence of a metric implies that the only way to contract indices is by the anti-symmetric epsilon tensor. In $(2+1)D$ this leaves only one option namely the Chern-Simons action (1.69). In $(1+1)D$, there is also only one choice namely

$$\Gamma[A_\mu] = \frac{e}{4\pi} \int d^2x\, \theta\, \varepsilon^{\mu\nu} F_{\mu\nu} \,, \tag{1.70}$$

which we will refer to as the $1d$ θ-term. This is the integral of the electric field strength, and is thus, as opposed to the Chern Simons term, fully gauge invariant, and it does not contribute to the equations of motion. The choice of the symbol θ, which indicates an angle, is not accidental as will be clear below.

The θ-term is different from the Chern-Simons term in $(2+1)D$ in that uniform fields do not induce any currents but only polarization. In the static case, polarization amounts to creating a dipole density that partially screens the the external electric field, $\mathbf{D} = (1+\chi_e)\,\mathbf{E}$, or equivalently it creates edge charges. To see how the topological term alters this, consider a line segment with endpoints at $x = x_\pm$. Choosing the gauge with $A_x = 0$, and assuming θ in the $1d$ θ-term, (1.70), to be constant gives,

$$\Gamma = -e\frac{\theta}{2\pi} \int dt\, [A_0(x_+,t) - A_0(x_-,t)] \;. \tag{1.71}$$

Varying Γ with respect to $A_0(x_\pm,t)$ gives the charge at $x_\pm$,

$$Q^\pm = \mp e\, \frac{\theta}{2\pi} \;, \tag{1.72}$$

where Q^+ and Q^- are the charges on the right and left ends of the line segment respectively. Thus the topological term adds a constant to the edge charge. Including the usual non-topological action $\sim \int d^2x\, \chi_e \mathbf{E}^2$ (*cf.* the $2d$ linear response action

(1.21)) for a dielectric gives

$$Q^{\pm} = \pm\chi_e eV \mp e\frac{\theta}{2\pi} , \tag{1.73}$$

where $V = A_0(x_+,t) - A_0(x_-,t)$ is the voltage difference between the left and right end.

We can now see why θ should be regarded as an angular variable. Making the shift $\theta \to \theta + k2\pi$, amounts simply to adding k unit charges at the ends of the wire which is a local effect that for instance can be due to impurities. A non integer value of $\theta/2\pi$, on the other hand, is a bulk polarization effect, and we now show that it differs from the usual polarization $\sim \chi_e$ in being quantized as long as certain symmetries are respected.

For this, we again look at a system with periodic boundary conditions, i.e., a circle. Now imagine that we adiabatically transform a system from a trivial insulator to an insulator with a non-trivial $1d$ θ-term (1.70). During this process, a charge, Q, will be transported around the circle and this charge is, as we will see, given purely by θ. This is directly related to the bulk polarization, since to create a polarization charge Q, it has to flow past every point except close to the edges where it accumulates.

Varying the $1d$ θ-term, with respect to A_μ, the current

$$j_x = e\frac{\partial_t\theta(t)}{2\pi} , \tag{1.74}$$

and the total charge that has been transported around when $\theta(t)$ is changed from θ_1 to the final value θ_2, is

$$Q = \int dt\, j_x = \int dt\, e\frac{\partial_t\theta(t)}{2\pi} = e\frac{\theta_2 - \theta_1}{2\pi} .$$

Now lets calculate this in a microscopic picture using a many body state $|\psi(t,\phi)\rangle$ that start out in the atomic limit, and then evolves adiabatically to some state $|\psi(t_2,\phi)\rangle$ (ϕ denotes the magnetic flux passing through the circle). A similar set of manipulations as those used to show that the Hall conductance is the first Chern number gives

$$j_x = \left\langle \partial_t\psi \middle| \partial_\phi\psi \right\rangle - \left\langle \partial_\phi\psi \middle| \partial_t\psi \right\rangle \overset{def.}{=} \mathscr{F}_{t\phi} . \tag{1.75}$$

Here $\mathscr{F}_{t\phi}$ is the Berry field strength of the fiber bundle with the cartesian product of the time-interval $[t_1,t_2]$ and the space of flux-values through the circle as base space and on-dimensional fibers spanned by $|\psi(t,\phi)\rangle$. The total charge passing through a point on the circle during the process is now obtained by integrating over t. Also, as in the case of the Hall conductance, we invoke locality to average over the flux (here meaning that the polarizability cannot depend on the flux through the circle). We get,

$$Q = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{t_1}^{t_2} dt\, \mathscr{F} = \int_0^{2\pi} d\phi\, \mathscr{A}_\phi(t_2) - \int_0^{2\pi} d\phi\, \mathscr{A}_\phi(t_1)\ , \tag{1.76}$$

where

$$\mathscr{A}_\phi(t) = \left\langle \psi \,\middle|\, \partial_\phi \psi \right\rangle - \left\langle \partial_\phi \psi \,\middle|\, \psi \right\rangle \tag{1.77}$$

is the ϕ component of the Berry connection of the just mentioned fiber bundle. We now refer to section 1.8.1, where the $1d$ Chern-Simons invariant is defined as,

$$CS_1[\mathscr{A}(t)] = \frac{1}{2\pi}\int_0^{2\pi} \mathscr{A}_\phi(t) d\phi\ , \tag{1.78}$$

and where it is shown that 2π times the exponent of this is a well defined basis independent property. At t_1 the state is just a product state of localized electrons, so we can choose a gauge where $\mathscr{A}_\phi(t_1) = 0$ and it follows that

$$e^{2\pi CS_1[\mathscr{A}(t_1)]} = 1 \qquad ; \qquad e^{i2\pi Q/e} = e^{2\pi CS_1[\mathscr{A}]}\ , \tag{1.79}$$

where t_2 is suppressed since the state $|\psi(t_2)\rangle$ is the many-body state of interest i.e., we define $\mathscr{A}_\phi \equiv \left\langle \psi(t_2) \,\middle|\, \partial_\phi \psi(t_2) \right\rangle - \left\langle \partial_\phi \psi(t_2) \,\middle|\, \psi(t_2) \right\rangle$.

The above phase (1.79) has an alternative interpretation as the Berry phase accumulated when a unit flux is adiabatically inserted through the circle. Since both time-reversal and chiral symmetry (that is charge conjugation composed with time reversal) maps inserting an upward flux to inserting a downward flux, one can conclude that if any of these symmetries are present during the adiabatic process, one has

$$e^{\int_0^{2\pi} \mathscr{A}_\phi d\phi} = e^{-\int_0^{2\pi} \mathscr{A}_\phi d\phi}\ , \tag{1.80}$$

which leaves only two possibilities,

$$e^{iQ/e} = e^{2\pi CS_1[\mathscr{A}]} = \begin{cases} e^{i\pi} & \text{non-trivial} \\ 0 & \text{trivial} \end{cases}\ , \tag{1.81}$$

corresponding to having $\theta = 0$ or $\theta = \pi \mod 2\pi$.

If we have a non-interacting system with lattice translation invariance we will as in 2+1 dimensions have a fiber bundle defined by the Bloch states over the Brillouin zone circle. Again, in the same way as the Chern number, the exponent of the Chern-Simons invariant of the flux-circle bundle will be the same as the exponent of the Chern-Simons invariant of the Brillouin zone circle. We will make use of this in the next section where we study a model which has a topological polarization response.

1.4.2 Realization with Dirac fermions

Rather than studying a lattice model, we shall pick a continuum model with a global chiral symmetry, and would therefore be expected to be characterized by the Chern-

Simons invariant and thus fall into one of the two classes we found above. Since topological response is a long distance effect, we expect that such a continuum theory will also describe chiral symmetric lattice models.

With this motivation, we shall investigate the $1d$ Dirac fermion ψ, with mass m, coupled to a gauge field a using path integral methods. Our starting point is the partition function,

$$Z[a] = \int \mathscr{D}[\bar{\psi}, \psi] e^{i\int d^2x\, \bar{\psi}(\gamma^\mu(i\partial_\mu - a_\mu) - m)\psi} . \tag{1.82}$$

We parametrize the gauge field as $a_\mu = \varepsilon_{\mu\nu}\partial_\nu \xi + \partial_\mu \lambda$, so that $F = \varepsilon^{\mu\nu}\partial_\mu a_\nu = -\partial^2 \xi$; the term containing λ is just a gauge transformation which does not contribute to the action (provided we are on a simply connected manifold). One can now verify that the *chiral transformation*

$$\psi \to e^{-i\gamma_3 \xi}\psi , \tag{1.83}$$

where $\gamma_3 = i\gamma_0\gamma_1$, eliminates the transverse gauge field $\varepsilon^{\mu\nu}\partial_\nu \xi$ from the action, while the mass term is changed,

$$\bar{\psi}(\gamma^\mu(i\partial_\mu - a_\mu) - m)\psi \to \bar{\psi}(\gamma^\mu i\partial_\mu - me^{-2i\gamma_3\xi})\psi . \tag{1.84}$$

This looks very strange, since for the massless case it seems like we by this transformation can get rid of a non-trivial field. The resolution of the apparent contradiction is that the path integral measure is not invariant under the transformation. Using techniques pioneered by Fujikawa [39], one can show that under the chiral transformation (1.83),

$$\mathscr{D}[\bar{\psi}, \psi] \to \mathscr{D}[\bar{\psi}, \psi] e^{-\frac{i}{2\pi}\int d^2x\, \xi \partial^2 \xi} , \tag{1.85}$$

which is the path integral incarnation of the axial anomaly referred to at the end of Section 1.2.4. In particular we shall be interested in a (space-time) constant chiral transformation $\xi(x) = -\theta/2$, which does not change the coupling to a_μ but only affects the mass term, and introduces a $1d$ θ-term, (1.70), in the response action.

For the case of the continuum Dirac equation we shall follow the same logic as in the 2+1 dimensional case, and only calculate how the value of θ *differs* between different phases. From (1.84) one realizes that taking $2\xi = \theta = \pi$ amounts to changing the sign of the fermion mass. Taking the gamma matrices,

$$\gamma^0 = \sigma^1 \qquad ; \qquad \gamma^1 = i\sigma^3 \qquad ; \qquad \gamma^3 = i\sigma^2 , \tag{1.86}$$

we have

$$H = \begin{pmatrix} 0 & m - ik \\ m + ik & 0 \end{pmatrix} = \begin{pmatrix} 0 & Q^\dagger \\ Q & 0 \end{pmatrix} . \tag{1.87}$$

It is straightforward to obtain the wave functions, and calculate the Brillouin-zone Berry potential,

$$\mathscr{A} = \frac{1}{2}\frac{m}{k^2+m^2}\,. \tag{1.88}$$

We can now form the correponding Chern-Simons invariant by integrating over the filled states labeled by k,

$$CS_1[\mathscr{A}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\mathscr{A}\ . \tag{1.89}$$

Using the expression for the Berry connection of the Brillouin-zone fiber bundle (1.88) and the definition of the Chern-Simons invariant (1.89) we get

$$CS_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\frac{1}{2}\frac{m}{k^2+m^2} = \frac{1}{4}\frac{m}{|m|} \tag{1.90}$$

for the filled Dirac sea. Previously, in the discussion of the Chern insulator, we saw that the Chern number of the Brillouin-zone fiber bundle and the flux-torus fiber bundle were equal. Analogously the just derived Brillouin zone Chern-Simons invariant equals the flux-circle Chern-Simons invariant, which was introduced in the previous section and was shown to be proportional to the topological polarisation.

There is an alternative way to characterize the topology of Hamiltonian of the form on the right hand side of (1.87). (It can be shown that a general Hamiltonian with chiral symmetry can be written in this form [5], so it applies to more than the Dirac equation.) This alternative characteristic is by the *winding number* defined by,

$$w = \frac{i}{2\pi}\int dk\,Q^{-1}\partial_k Q = \frac{i}{2\pi}\int dk\,\frac{-im}{k^2+m^2} = \frac{1}{2}\frac{m}{|m|}\,, \tag{1.91}$$

i.e., it equals twice the invariant CS_1. Note that the winding number changes by one unit when the sign of the mass changes, consistent with θ in (1.70) changing by π.

Just as in the discussion of the Chern number for the Dirac sea, you might wonder how something that is called a winding number can be non-integer. The resolution is again related to the regularization of the continuum Dirac theory. If we instead consider the lattice version,

$$H_{lat} = \sin(k)\sigma^2 + (m-1+\cos(k))\sigma^1 \tag{1.92}$$

so that $Q = -i\sin(k) - (m-1+\cos(k))$ we get

$$w = \frac{i}{2\pi}\int_{\pi}^{\pi} dk\,\partial_k \ln(m-1+e^{-ik})\,. \tag{1.93}$$

For $m<0$ the curve $m-1+e^{-ik}$ does not wind around the origin, so the logarithm can be picked to be single valued and thus $w=0$. For $0<m<2$ it winds one turn in the negative direction and $w = 1$. The change of the winding at $m = 0$ is the same as in the continuum model.

1.4.3 Higher dimensions

Both in the $2+1$ and the $1+1$ dimensional cases, there is only one possible topological $U(1)$ symmetric response term. This is the case in any dimension, and it turns out that all odd space time dimension cases are similar to the $(2+1)D$ case while all even ones are similar to the $(1+1)D$ case.

To find topological response terms in D space-time dimension we need a D-form to contract with the anti-symmetric epsilon tensor with a result that is gauge invariant in the bulk. These conditions are very restrictive, and leave us with,

$$\Gamma[A_\mu] \propto \int d^2x\, \varepsilon^{\mu\nu} F_{\mu\nu} \qquad D = 2 \tag{1.94}$$

$$\Gamma[A_\mu] \propto \int d^3x\, \varepsilon^{\mu\nu\sigma} A_\mu F_{\nu\sigma} \qquad D = 3 \tag{1.95}$$

$$\Gamma[A_\mu] \propto \int d^4x\, \varepsilon^{\mu\nu\sigma\lambda} F_{\mu\nu} F_{\sigma\lambda} \qquad D = 4 \tag{1.96}$$

$$\Gamma[A_\mu] \propto \int d^5x\, \varepsilon^{\mu\nu\sigma\lambda\kappa} A_\mu F_{\nu\sigma} F_{\lambda\kappa} \qquad D = 5 \tag{1.97}$$

$$\vdots \qquad\qquad\qquad \vdots$$

You notice a difference between even and odd space time dimensions. In the even case the actions are Chern-Simons terms that only are gauge invariant up to edge terms. The most important example is the already discussed $2d$ case, and the higher dimensional analogs are very similar:

Using similar arguments to the ones for the $(2+1)D$ case one can show that the response in $D = 2k+1$ following from the response above (1.97) is the k^{th} Chern number of the flux-torus bundle and for a non-interacting system in a periodic potential this equals the k^{th} Chern number of the bundle of filled states over Brillouin zone (there are $2k$ independent fluxes that one can thread in a $d = 2k$ dimensional torus). Thus, just as in $2d$, there is a quantized current response.

Using similar arguments as in the $(1+1)D$ case one can show that the response in $D = 2k$ is given by the exponent of 2π times the k^{th} Chern-Simons invariant of the flux torus bundle. This is as in $(1+1)D$, not quantized unless one can assume either time-reversal or chiral symmetry.

Let us briefly mention the most important example of these topological responses, namely the one in $(3+1)D$ that describes a $3d$ time-reversal invariant topological insulator:

$$\Gamma[A_\mu] = \frac{\theta\sigma_0}{16\pi}\int d^4x\,\varepsilon^{\mu\nu\sigma\lambda}F_{\mu\nu}F_{\sigma\lambda} = \frac{\theta\sigma_0}{2\pi}\int d^4x\,\mathbf{E}\cdot\mathbf{B}\,.$$

This term has many important implications that we will list without any derivations:

- A time-reversal invariant system with $\theta = \pi \mod 2\pi$, i.e. a $3d$ topological insulator, has a gapless surface mode.
- If the time-reversal invariance is weakly broken at the surface, the system will have a quantized Hall response with $\sigma_H = \frac{1}{2}\sigma_0 \mod \sigma_0$ [40].
- The *Witten effect* [41]: A magnetic monopole will be accompanied by a localized electric charge of $\frac{1}{2}e \mod e$, and conversely, putting an electric charge near the surface of a topological insulator, will induce a "magnetic monopole" in the bulk! [42].

1.5 Systems characterized only by edge modes

The topological phases we have considered so far were all characterized by their $U(1)$ response. All of them also supported gapless edge modes; so, you might think that these characteristics go hand in hand. As we shall now discuss, this is not true: there are topological phases of matter which have no topological response, but still exhibit topologically protected gapless edge modes. We shall exemplify with two of the most important cases: the $2d$ time-reversal invariant topological insulator, and the Kitaev chain with and without time-reversal invariance, which is also referred to as a $1d$ topological superconductor.

1.5.1 The time-reversal invariant topological insulator

Although you learned that a magnetic field is not necessary for having a quantized Hall response, clearly time-reversal invariance has to be broken since the current is flowing in a particular direction. This invites the question of whether it is possible to have topologically non-trivial states which are time-reversal invariant. An obvious way to get such a system is to add two copies of a Chern insulator, and since the particles are electrons, the natural candidates for the two "species" are the two spin directions: up and down. Such a system will exhibit a quantized *spin Hall effect* that can be described very similarly to the Chern insulator. Just as in case of the integer QH effect, we can construct a topological field theory to describe the quantum spin Hall effect. Since there are now two conserved currents, corresponding to spin up and spin down, we expect a topological action with two gauge fields $b^\uparrow$ and $b^\downarrow$.

$$\mathscr{L}_{QSH} = -\frac{1}{4\pi}\left(\varepsilon^{\mu\nu\sigma}b^\uparrow_\mu\partial_\nu b^\uparrow_\sigma + \varepsilon^{\mu\nu\sigma}b^\downarrow_\mu\partial_\nu b^\downarrow_\sigma\right). \tag{1.98}$$

This is an example of a *doubled Chern-Simons theory*. There is no QH effect, since the contributions to σ_H obtained by coupling to an external electromagnetic field come with different signs and cancel each other. Similarly, there is no chiral electric edge current, but instead a chiral spin current. This is however only relevant in the cases where one component of the spin is conserved, which is normally not the case in real materials. Surprisingly, however, there are topologically distinct states even when the spin current is not conserved. It is these time-reversal invariant states that are commonly referred to as topological insulators. To arrive at this conclusion using field theory would require that we break the symmetry as $U(1) \times U(1) \to U(1)$, and show that the resulting theory still has protected gapless edge modes. As of now, we do not know of any way to do this. However, since the seminal work by Kane and Mele [43, 44], it has been known that the gapless edge modes persists even when the symmetry is broken. Furthermore they showed that there is new topological "Z_2 invariant" that tells whether the insulator is trivial with no edge modes, or topological with gapless edge modes. The analysis by Kane and Mele was for non-interacting systems, and, at least in this limit, the topological insulator is a time-reversal protected SPT state. The edge modes are seen in many experiments with topological insulators, and are believed to be a generic feature of a large class of materials, see e.g., [3].

1.5.2 The Kitaev chain

All phases we have discussed so far have a $U(1)$ symmGennestry corresponding to conservation of electric charge. There are, however, important phases of matter which are most easily described using a formalism where this symmetry is broken. This is when there is a condensate, either of fundamental bosons (like in ^{4}He), or Cooper pairs (as in a superconductor) that act as a reservoir of charged particles. Mathematically this amounts to having terms of the type $\sim \Delta\psi\psi$ in the Hamiltonian, and in this section we will consider the "reservoir" Δ as static, and the electromagnetic fields as a background with no dynamics. An important physical situation where this happens realized is when Δ is generated by proximity to a superconductor.

We will refer to quadratic Hamiltonians with terms breaking $U(1)$ as Bogolliobov–de Gennes (BdG) Hamiltonians. It has been realized for quite some time [4, 5] that BdG systems are also part of the comprehensive classification schemes of free fermion theories that was alluded to in the introduction.

Of particular interest are the chiral systems of p-wave type, since their boundary theories typically support *Majorana* modes which can be thought of as a "half fermion ". This is in fact one of the most striking examples of edges of topological phases that cannot be realized as *bona fide* boundary systems not connected to a bulk. The simplest example of this is the Kitaev chain, which is of large current interest since it can be experimentally realized in quantum wires with strong spin-orbit coupling [45] or in chains of magnetic atoms on top of a superconductor [46].

The following presentation is a shortened, and somewhat simplified, adaption of [9]. The *Kitaev chain* is a model for spinless fermions hopping on a $1d$ lattice, and is given by the Hamiltonian

$$H_K = \sum_j \left[-t(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) - \mu(a_j^\dagger a_j - \tfrac{1}{2}) + \Delta^\star a_j a_{j+1} + \Delta a_{j+1}^\dagger a_j^\dagger \right] . \quad (1.99)$$

Here t is a hopping amplitude, μ a chemical potential, and Δ an induced superconducting gap. In terms of the Majorana fields,

$$c_{2j-1} = a_j + a_j^\dagger \qquad ; \qquad c_{2j} = \frac{1}{i}(a_j - a_j^\dagger) , \quad (1.100)$$

the Hamiltonian becomes

$$H_K = \frac{i}{2} \sum_j \left[-\mu c_{2j-1} c_{2j} + (t+\Delta) c_{2j} c_{2j+1} + (-t+\Delta) c_{2j-1} c_{2j+2} \right] . \quad (1.101)$$

Let us now consider two special cases.

1. The trivial case: $\Delta = t = 0,\ \mu < 0$.

$$H_1 = -\mu \sum_j (a_j^\dagger a_j - \frac{1}{2}) = \frac{i}{2}(-\mu) \sum_j c_{2j-1} c_{2j} \quad (1.102)$$

   The Majorana operators c_{2j-1}, c_{2j} related to the fermion ψ_j on the site j are paired together to form a ground-state with the occupation number 0.
2. $\Delta = t > 0,\ \mu = 0$. In this case

$$H_2 = it \sum_j c_{2j} c_{2j+1} , \quad (1.103)$$

   and here the Majorana operators c_{2j}, c_{2j+1} from different sites are paired. We can define new annihilation and creation operators

$$\tilde{a}_j = \frac{1}{2}(c_{2j} + i c_{2j+1}) \qquad ; \qquad \tilde{a}_j^\dagger = \frac{1}{2}(c_{2j} - i c_{2j+1}) , \quad (1.104)$$

   which are shared between sites j and $j+1$. The Hamiltonian becomes

$$H_2 = 2t \sum_{j=1}^{L-1} (\tilde{a}_j^\dagger \tilde{a}_j - \frac{1}{2}) . \quad (1.105)$$

Note that neither c_1 nor c_{2L} is part of the Hamiltonian, and as a consequence the ground-state is degenerate, since these two Majorana operators can be combined to a fermion ,

$$\Psi = \frac{i}{2}(c_1 + i c_{2L}) , \quad (1.106)$$

which can be occupied or unoccupied, corresponding to a two-fold degeneracy. The fermion number corresponding to this field, is, however, delocalized at the two ends of the chain. Thus, no local perturbation can change the occupation of this state. It is this property that has made fractionalized fermions an interesting object for quantum information technology. *If a qubit could be stored in a pair of spatially separated Majorana fermions, it would be very robust against noise* [47].

Although the above analysis was only for two very special points in the parameter space, Kitaev established that the whole parameter region $\Delta \neq 0$, and $|\mu| < 2|t|$ is topological.

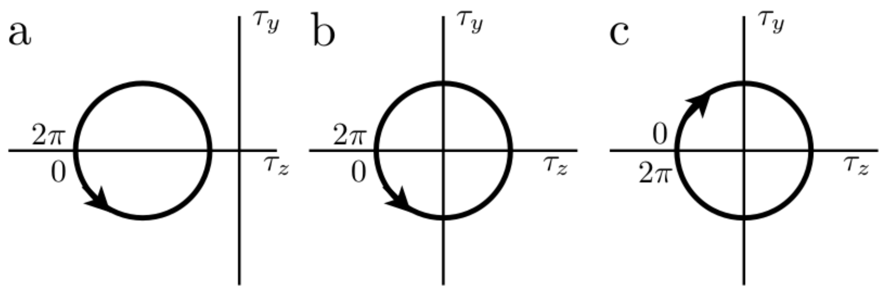

Fig. 1.4 Winding numbers ν of $\mathbf{d}(k)$ for the full Kitaev chain, in (a) trivial phase with $w = 0$, for $0 < t < \mu/2$, $\Delta > 0$, (b) topological phase with $w = 1$ for $\mu = 0$, $0 < t = \Delta$ and (c) topological phase with $w = -1$ for $\mu = 0$, $0 < t = -\Delta$. The arrows denote the direction in which k increases.

It is illustrative to see how a topological index appears if we assume $\Delta \in \mathbb{R}$, which corresponds to time-reversal invariance. To this end, we introduce the Nambu spinor $\Psi_k^\dagger = (a_k^\dagger, a_{-k})$ in terms of which the Hamiltonian (1.99) in momentum space becomes,

$$H_K = \sum_k \Psi_k^\dagger \mathscr{H}_K(k) \Psi_k \;, \tag{1.107}$$

with $\mathscr{H}_K(k)$ given by

$$\mathscr{H}_K(k) = \big(-\mu/2 - t\cos(k)\big)\tau_z - \Delta \sin(k)\tau_y = -\mathbf{d}(k)\cdot\tau \tag{1.108}$$

where the Pauli-matrices τ_i act in the particle-hole spinor space. Just as in the case of the $1d$ insulator discussed, the topological invariant takes the form of a *winding number*. To show this, consider the curve traced out by the vector $\mathbf{d}(k) = (0, \Delta \sin(k), \mu/2 + t\cos(k))$ in the (τ_y, τ_z)-plane (i.e., the space of Hamiltonians), as k sweeps through the full Brillouin zone. This is illustrated in Fig. 1.4, where we schematically show the curve swept by $\mathbf{d}(k)$ in the trivial phase, with winding $w = 0$, and the two different topological phases, with winding $w = \pm 1$.

One can construct Hamiltonians that realize any integer number of windings (a simple way is just to take many copies of the Kitaev wire), and one would think that there is one topologically distinct state for every integer, a so called $\mathbb{Z}$ classification.

However, it turns out that this is true only in the absence of interaction. Fidkowski and Kitaev [48] showed that including carefully chosen four-fermion interactions, there are only eight distinct phases which amounts to a $\mathbb{Z}_8$ classification.

Note that the result so far is based on time-reversal invariance symmetry that ensures that Δ is real. So, the eight states are time-reversal protected SPT states. If the time-reversal symmetry is broken (e.g. by a current in the s-wave superconductor inducing Δ) the phase of Δ will vary, and the winding number will no longer be well defined. However, the Majorana modes are still present for general Δ, but there is only a $\mathbb{Z}_2$ index—either there is a Majorana or not[10]. The Majorana modes are present even in the presence of interactions, and no symmetry was assumed; So, this is an example of TO, but of the kind, mentioned in the introduction, that resembles SPT states. One can thus regard the Majorana chain as being protected by fermion parity [10], which is not a symmetry but a property common to all fermionic systems.

1.6 Superconductors are topologically ordered

In this section we shall discuss the perhaps simplest example of a topologically ordered state, namely an usual s-wave paired BCS superconductor coupled to electromagnetism described by Maxwell theory. We start by some general remarks and then turn to examples.

It is known that even a weak attractive interaction will turn a Fermi liquid into a superconductor at sufficiently low temperature. The mechanism is the formation of Cooper pairs composed of two electrons with equal but opposite momenta. Such pairs form, not because of the strength of the interaction, but because of the large available phase space, given by the Fermi surface. This means that even though the theory is weakly coupled (in conventional superconductors by an electron-phonon interaction) the ground-state is non-perturbative.

Most common superconductors have Cooper pairs where the spins form a singlet, which forces the orbital wave function to be symmetric. The simplest possibility is an *s*-wave, and this is in fact the symmetry of the order parameter in most conventional, or low T_c, superconductors. From the quantum field theory point of view, the BCS approach to superconductivity amounts to a self-consistent mean field approximation, based on the pairing field $\Delta(x) = \psi_\uparrow(x)\psi_\downarrow(x)$.

The fermions are now gapped and can be integrated out. The resulting field theory is to a good approximation the Ginzburg-Landau theory for superconductors.

The Ginzburg-Landau theory supports vortex solutions. The elementary vortex has a core that captures half a unit of magnetic flux, since the Cooper pair has charge $2e$. This means that a quasi-particle will pick up a phase $e^{i\pi} = -1$ when en-

[10] This has a direct consequence for the spectrum of Josephson junctions. In the first (real) case, the junction between two topological states with winding number ± 1, will host two Majorana zero-modes, which amounts to a single Dirac zero-mode, while in the second (complex) case such a junction will have no zero-mode, see e.g., [49].

circling a vortex at a distance large enough for it not to penetrate the vortex core. This is an example of a *topological braiding phase*, which can, as we shall show, readily be captured by a topological field theory. We will now clarify the distinction between a real, "fluctuating", superconductor coupled to electromagnetism, and a model superconductor described by a BdG theory without a dynamical electromagnetic field. The difference, which was first clearly pointed out by [50], is that the former totally screens the electromagnetic current while the second does not. Thus, in the fluctuating superconductor, the *only* low energy degrees of freedom are the vortices, and electrically neutral fermionic quasi-particles. That this is a topologically ordered phase was first pointed out by [51]. The topological field theory that describes the low-energy properties of the s-wave superconductor is the very same BF theory that we already discussed in connection with the IQHE and the Chern insulator. For simplicity we will focus on the $2d$ case, but with a short comment on the $3d$ case in Sect. 1.6.2. For pedagogical reasons we shall first give a heuristic derivation of the BF theory, using an analogy with the Chern-Simons theory discussed in Sect. 1.2.3. Later, in Sect. 1.6.3, we outline a derivation of the BF theory from a microscopic model.

1.6.1 BF theory of s-wave superconductors—heuristic approach

We first consider the $(2+1)D$ case where both quasi-particles and vortices are particles, and we can proceed in close analogy to the bosonic Chern-Simons theory for the quantum Hall effect. Recall that in that case the equations of motion relating charge and flux, and the statistics of the quasi-particles (which in that case is simply holes in the filled Landau level) followed from the coupling to the gauge field. The present case differs from the above in that we have two distinct excitations, quasi-particles and vortices, and we will describe them with two conserved currents, j_q^μ and j_v^μ, which we couple to two different gauge fields, a and b, by the Lagrangian,

$$\mathscr{L}_{curr} = -a_\mu j_q^\mu - b_\mu j_v^\mu \, . \tag{1.109}$$

A simple calculation shows that in order to get a phase π when moving a j_q quantum around a j_v quantum we should take

$$\mathscr{L}_{BF} = \frac{1}{2\pi} \varepsilon^{\mu\nu\sigma} b_\mu f^{(a)}_{\nu\sigma} \, , \tag{1.110}$$

where $f^{(a)}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. This we recognize as the BF action, but with a coefficient that differs from (1.42) derived in the QH context. Putting the parts together we have the topological action,[11]

[11] The symmetry properties of the Lagrangian (1.111) are worth a comment. Under the parity transformation $(x,y) \to (-x,y)$ the two potentials transform as $(a_0,a_x,a_y) \to (a_0,-a_x,a_y)$ and $(b_0,b_x,b_y) \to (-b_0,b_x,-b_y)$, while under time reversal the transformations are, $(a_0,a_x,a_y) \to$

$$\mathscr{L}_{top} = \frac{1}{\pi}\varepsilon^{\mu\nu\sigma} b_\mu \partial_\nu a_\sigma - a_\mu j_q^\mu - b_\mu j_v^\mu \,. \tag{1.111}$$

The topological nature of $\mathscr{L}_{top}$ is clear from the equations of motion,

$$j_v^\mu = \frac{1}{\pi}\varepsilon^{\mu\nu\sigma}\partial_\nu a_\sigma = \frac{1}{2\pi}\varepsilon^{\mu\nu\sigma} f^{(a)}_{\nu\sigma} \quad ; \quad j_q^\mu = \frac{1}{\pi}\varepsilon^{\mu\nu\sigma}\partial_\nu b_\sigma = \frac{1}{2\pi}\varepsilon^{\mu\nu\sigma} f^{(b)}_{\nu\sigma} \,, \tag{1.112}$$

which show that the gauge invariant field strengths are fully determined by the currents, just as in the Chern-Simons theory. These equations both have a very direct physical interpretation. For instance, if we write (1.112) as $j_q^\mu + (1/\pi)\varepsilon^{\mu\nu\sigma}\partial_\nu b_\sigma = 0$ this expresses that the quasi-particle current is totally screened by the superconducting condensate shows that $(-1/\pi)\varepsilon^{\mu\nu\sigma}\partial_\nu b_\sigma$ should be interpreted as the screening current. This observation can indeed be used to give an alternative derivation, or rather motivation, for our topological field theory; the potential a_μ is nothing but a Lagrange multiplier that enforces the constraint of total screening of the current j_q. For a more detailed discussion, see [18].

It is interesting to consider the quantization and conservation of charge in our topological field theory. Quantisation of charge implies that the current couples to a $U(1)$ gauge field A_μ that is not only invariant under

$$A_\mu \to A_\mu + \partial_\mu \Lambda \,, \tag{1.113}$$

where Λ is a continues well-defined (i.e. single valued) function, but the more general transformation

$$A_\mu \to A_\mu + \frac{1}{q} e^{i\Lambda} i\partial_\mu e^{-i\Lambda} \,, \tag{1.114}$$

where q is the minimal charge, and $e^{i\Lambda}$ is a well defined function, although Λ is not a continues well-defined function. We call a gauge field with this property *compact*. A simple example will clarify the distinction between compact and non-compact gauge filelds. Consider a $2d$ gauge field and use polar coordinates (ρ,φ). An usual gauge transformation (1.113) with $\Lambda = n\varphi$ is not allowed since Λ is not a continues well-defined function. However, $e^{in\varphi}$ is, and the gauge transformation (1.114) reads $A_\varphi \to A_\varphi + n/q\rho$. Although this transformation introduces a singularity in the gauge field, it leaves the Wilson loop $\exp(iq\oint A_\varphi)$, for any closed curve (including one that encircles the singularity at the origin) invariant, and you can think of the gauge transformation as inserting an invisible flux in the system.

If we want the two currents j_v^μ and j_q^μ to be integer valued, the above argument leads us to require the gauge fields a_μ and b_μ to be compact and transform as

$$a_\mu \to a_\mu + e^{i\Lambda_a} i\partial_\mu e^{-i\Lambda_a} \quad ; \quad b_\mu \to b_\mu + e^{i\Lambda_b} i\partial_\mu e^{-i\Lambda_b} \,. \tag{1.115}$$

$(a_0, -a_x, -a_y)$ and $(b_0, b_x, b_y) \to (-b_0, b_x, b_y)$, respectively. The unusual transformation properties of the potential b_μ follow from those of the vortex current. It is easy to check that the *BF* action is invariant under both P and T.

The question of current conservation is more subtle. Since the world line of a point-like vortex can be thought of as a vortex line in space-time, non-conservation of the vortex charge would amount to having such world lines ending at a point. This could happen if there were unit charge magnetic monopoles in space-time, on which two such world lines could terminate. Such monopoles in space-time are called *instantons* and are known to exist in many field theories. Since we do not have any magnetic monopoles, this is however not a realistic option, and the vortex charge is conserved. The situation is quite different when it comes to the electric charge. Here the Cooper-pair condensate acts as a source of pairs of electrons, and in our topological theory such processes, corresponding to formation or breaking of pairs, this could be incorporated by having instantons in the b field.

1.6.2 The 3+1 dimensional BF theory

Turning to the case of 3+1 dimensions, we have essentially the same construction, but with the difference that the vortices are now strings, and the field b is an anti-symmetric tensor, $b_{\mu\nu}$. The action is still again of the BF type and reads,

$$\mathscr{L}_{BF} = \frac{1}{\pi}\varepsilon^{\mu\nu\sigma\lambda} b_{\mu\nu}\partial_\sigma a_\lambda \,. \tag{1.116}$$

The gauge transformations of the b field are given by

$$b_{\mu\nu} \to b_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \ , \tag{1.117}$$

where ξ_μ is a vector-valued gauge parameter. The minimal coupling of the b potential to the world sheet, Σ, of the string is given by the action,

$$S_{vort} = -\int_\Sigma d\tau d\sigma^{\mu\nu} b_{\mu\nu} = -\int_\Sigma d\tau d\sigma \left| \frac{d(x^\mu, x^\nu)}{d(\tau,\sigma)} \right| b_{\mu\nu} \,, \tag{1.118}$$

where (τ,σ) are time and space like coordinates on the worldsheet. This is a direct generalization of the coupling of a to the world line, Γ, of a spinon,

$$S_{sp} = -\int_\Gamma dx^\mu a_\mu = -\int_\Gamma d\tau \frac{dx^\mu}{d\tau} a_\mu \,. \tag{1.119}$$

Combining these elements we get the topological action for the 3+1 dimensional superconductor,

$$S_{top} = \int d^4x \mathscr{L}_{BF} + S_{sp} + S_{vort} \,. \tag{1.120}$$

The proof that this action indeed gives the correct braiding phases can be found e.g., in [31], and a discussion of this action in the context of superconductivity has appeared before in [52].

1.6.3 Microscopic derivation of the BF theory

So far, we did not derive the BF theory, but rather constructed it, or guessed it, from the the braiding properties of quasi-particles and vortices. It would obviously be more reassuring if the theory could be *derived* from a microscopic model. Here we outline such a derivation starting not from the original fermionic theory, but from an effective Ginzburg-Landau model coupled to a quasi-particle source. (The derivation of this theory from a model with paired electrons is a standard exercise that can be found e.g. in the book by [12].) For simplicity, we shall follow [18] and consider a toy version of the Ginzburg-Landau theory, namely the $(2+1)D$ relativistic Abelian Higgs models defined by the Lagrangian,[12]

$$\mathscr{L}_{AH} = \frac{1}{2M}|iD_\mu \phi|^2 - \frac{\lambda}{4}(\phi^\dagger \phi)^2 - \frac{m^2}{2}\phi^\dagger \phi - \frac{1}{4}F_{\mu\nu}^2 - eA_\mu j_q^\mu \ , \tag{1.121}$$

where we used a standard particle physics notation where ϕ is the charge $-2e$ scalar field representing the Cooper pair condensate, $iD_\mu = i\partial_\mu + 2eA_\mu$ is the covariant derivative, $F_{\mu\nu}$ is the electromagnetic field strength and the conserved current j_q^μ, with charge e, is introduced to describe the gapped quasi-particles discussed above.[13] Just as in the usual Ginzburg-Landau theory the Abelian Higgs model supports vortex solutions, which are characterized by a singularity in the phase of the Cooper-pair field. Defining $\phi = \sqrt{\rho}\, e^{i\varphi}$, and writing $\varphi = \tilde{\varphi} + \eta$, where η is regular, and can be removed by a proper gauge transformation, the vorticity is encoded in the gauge field $a_\mu = \frac{1}{2}\partial_\mu \varphi$, which depends on the vortex positions. The corresponding conserved vortex current is parametrized as $j_v = (1/\pi)\varepsilon^{\mu\nu\sigma}\partial_\mu a_\sigma$. It is now a matter of algebraic manipulations, involving integrating out gapped degrees of freedom, to derive the topological theory (1.111), see [18]. For this to work, it is crucial that we include a dynamical electromagnetic field, it is only then that the external currents are completely screened and all bulk modes are gapped and can be integrated over. Making a derivative expansion, and keeping terms to second order results in the effective Lagrangian,

---

[12] This is a toy model not only because we use a relativistic form for the kinetic energy, but also because we use $2+1D$ Maxwell theory, which amounts to a logarithmic Coulomb interaction. The generalization to the more realistic case is straightforward, and the result is qualitatively the same. The derivation, however, becomes less transparent. For the interested reader [53] is a good reference to see how to include fermions and also see how the chiral d-wave case works.

[13] Note that in spite of the relativistic form we normalize the kinetic term such that $|\phi|^2$ has the dimension of density.

$$\mathscr{L}_{eff} = \frac{1}{\pi}\varepsilon^{\mu\nu\sigma} b_\mu \partial_\nu a_\sigma - \frac{1}{4e^2}(f^{(a)}_{\mu\nu})^2 - \frac{1}{4}\left(\frac{e}{m_s \pi}\right)^2 (f^{(b)}_{\mu\nu})^2 - a_\mu j^\mu_q - b_\mu j^\mu_v \,. \tag{1.122}$$

where $m_s^2 = 4e^2\bar{\rho}/M$, and $\bar{\rho}$ is the average density. Note that the topological theory emerges as the leading term in this expansion! The higher derivative terms, which are of Maxwell form, are not topological, and have the effect of introducing bulk degrees of freedom. These are however gapped, and can be identified as the plasmon mode. At low energies the plasmons can be neglected and we can retain only the topological part. Another physical effect captured by the Maxwell terms is the London penetration length λ_L which is the size of the magnetic flux tube associated to a vortex. In the purely topological theory, the vortices are strictly point-like. In this section we strictly dealt with s-wave superconductors. The extension to the d-wave case is relatively straightforward, but since there are gapless quasi-particles associated to the nodal lines, the effective theory must include these, and becomes quite a bit more complicated [54]. The p-wave case is considerably more difficult because of the Majorana modes associated to vortices.

1.6.4 The two-dimensional p-wave superconductor

In the previous section we assumed that the fermionic part of the Lagrangian was gapped and could be integrated out. In the p-wave case this cannot be done since there are Majorana modes associated with vortices. These modes form a finite degenerate subspace of the Hilbert space, and the system has a most amazing feature: Quasi-adiabatic braidings of the vortices correspond to non-commuting unitary operators acting on the finite-dimensional Hilbert space. Since the vortices should are identical particles, this amounts to having non-Abelian fractional statistics. There is a very close resemblance between this state and the strongly correlated Moore-Read pfaffian quantum Hall state [55], that is a candidate wave function for the plateau observed at filling fraction $\nu = 5/2$. (This state will be discussed briefly in the last section.)

The simplest realization of a $2d$ chiral p-wave superconductor is by spin-less electrons with a parabolic dispersion relation, i.e., taking the single-particle Hamiltonian to be $H_1 = (-i\nabla - e\mathbf{A})^2/2m$ and adding an attractive short-range interaction,

$$\hat{V} = \frac{\lambda}{2}\int d^2x d^2y \left[\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r})\nabla^2\delta^2(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r})\psi(\mathbf{r}')\right]_{reg} \,, \tag{1.123}$$

where *reg* denotes that the potential can be approximated by $\lambda\nabla^2\delta^2(x-y)$ only for small momentum transfers close to the fermi surface. (The precise form of the potential is not needed for the coming discussion.)

Since we expect a condensation of Cooper pairs, it is practical to introduce a Hubbard-Stratonovich boson, $\mathbf{\Delta}$, which mediates the interaction. More precisely, we add an action for $\mathbf{\Delta}$ so that the classical solution is $\mathbf{\Delta}_{cl} = \lambda(\psi\nabla\psi)_{reg}$, meaning that we can, at the mean-field level, replace the quartic interaction by a term $\mathbf{\Delta}\cdot\psi^{\dagger}\nabla\psi^{\dagger} + h.c.$ ($h.c.$ means Hermitean conjugate).

To proceed we assume a chiral condensate and make the mean-field ansatz $\Delta \equiv \Delta_{\bar{z}} = \lambda\langle\psi\partial_{\bar{z}}\psi\rangle_{reg}$ and $\Delta_z = \lambda\langle\psi\partial_z\psi\rangle_{reg}$=0. For simplicity we will from now on drop the subscript *reg,* but all momentum sums involving Δ should be understood to have a cutoff. With these assumptions we get the Hamiltonian,

$$H = \int d^2x\ \psi^{\dagger}\left(H_1 - \mu\right)\psi + \psi^{\dagger}\Delta\partial_z\psi^{\dagger} + h.c.\,, \tag{1.124}$$

and proceed to find a homogeneous solution $\Delta = const$. Fourier transforming the field operator, $\psi(\mathbf{r}) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} c_{\mathbf{k}}/\sqrt{V}$, we get

$$H = \frac{1}{2}\sum_{\mathbf{k}}\left(c_{\mathbf{k}}^{\dagger}\ c_{-\mathbf{k}}\right)\underbrace{\begin{pmatrix}\xi_{\mathbf{k}} & \Delta\left(k_y + ik_x\right)\\ \Delta\left(k_y - ik_x\right) & -\xi_{\mathbf{k}}\end{pmatrix}}_{H_{\mathbf{k}}}\begin{pmatrix}c_{\mathbf{k}}\\ c_{-\mathbf{k}}^{\dagger}\end{pmatrix},$$

where $\xi_{\mathbf{k}} = \mathbf{k}^2/2m - \mu$. From the property

$$\sigma_x H_{\mathbf{k}}^{*}\sigma_x = -H_{-\mathbf{k}}\,, \tag{1.125}$$

it follows that all negative-energy solutions at $-\mathbf{k}$ can be written in terms of positive-energy solutions $H_{\mathbf{k}}(U_{\mathbf{k}},V_{\mathbf{k}})^T = E_{\mathbf{k}}(U_{\mathbf{k}},V_{\mathbf{k}})^T$, at $\mathbf{k}$. We can thus diagonalize the Hamiltonian with the positive-energy solutions $\Gamma_{\mathbf{k}}^{\dagger} = U_{\mathbf{k}}c_{\mathbf{k}}^{\dagger} + V_{\mathbf{k}}c_{-\mathbf{k}}that$,

$$H = E_0 + \sum_{\mathbf{k}} E_{\mathbf{k}}\Gamma_{\mathbf{k}}^{\dagger}\Gamma_{\mathbf{k}}\ ,$$

and the positive eigenvalues are obtained by directly diagonalizing H_k,

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta\mathbf{k}|^2}\ .$$

Note that the normalization $|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$ follows from requiring $\left\{\Gamma_{\mathbf{k}}^{\dagger},\Gamma_{\mathbf{k}'}\right\} = \delta_{\mathbf{k}\mathbf{k}'}$. The ground-state ($|BCS\rangle$) is thus the state annihilated by all $\Gamma_{\mathbf{k}}$, i.e., $\propto \prod_{\mathbf{k}}\Gamma_{\mathbf{k}}|0\rangle$. To get the normalization right, we note that

$$\Gamma_{\mathbf{k}}\Gamma_{-\mathbf{k}}|0\rangle = V_{-\mathbf{k}}^{*}\left(U_{\mathbf{k}}^{*} + V_{\mathbf{k}}^{*}c_{-\mathbf{k}}^{\dagger}c_{\mathbf{k}}^{\dagger}\right)|0\rangle\ ,$$

and from the requirement $|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$, we realize that the correctly normalized ground-state is

$$|BCS\rangle = {\prod_{\mathbf{k}}}'\left(U_{\mathbf{k}}^{*} + V_{\mathbf{k}}^{*}c_{-\mathbf{k}}^{\dagger}c_{\mathbf{k}}^{\dagger}\right)|0\rangle\ ,$$

where $\prod'$ indicates that the product is over half of the $\mathbf{k}$ values, as for instance,

$$\mathbf{k} \in \left\{ k_x > 0 \text{ or } k_x = 0,\ k_y \geq 0 \right\} \ . \tag{1.126}$$

To show that the ansatz $\Delta_z = 0$ and $\Delta_{\bar{z}} = constant$ is self-consistent, we calculate the mean field $\Delta_{x\pm iy}$,

$$\Delta_{x\pm iy} = \lambda \langle \psi \partial_{x\pm iy} \psi \rangle = \sum_{\mathbf{k}} (k_y \pm ik_x) U_{\mathbf{k}} V_{\mathbf{k}}^* = \sum_{\mathbf{k}} f(|\mathbf{k}|, \Delta)(k_x^2 \pm k_y^2) \ , \tag{1.127}$$

where f is a function determined by $U_{\mathbf{k}} V_{\mathbf{k}}^*$. In the last step we used $(U_{\mathbf{k}}, V_{\mathbf{k}}) \propto (f(|\mathbf{k}|, \Delta)(k_y + ik_x), 1)$ which is obtained by explicitly diagonalizing $H_{\mathbf{k}}$. Since the above sum is symmetric in interchange of k_x and k_y we infer that $\Delta_z = 0$. For Δ we instead get a gap-equation which determines its ground-state value.

We are now ready to see how the Majorana modes appear in the vortices. In a vortex solution the phase of Δ winds around some points (vortex cores). Close to a vortex core $\mathbf{r}'$ we have

$$\Delta_z(\mathbf{r}') = 0 \qquad ; \qquad \Delta(\mathbf{r}) = |\Delta|\, e^{in \arg(\mathbf{r}-\mathbf{r}') + i\lambda} \qquad n \in \mathbb{Z} \ ,$$

where λ is a regular function. There are no analytical self-consistent solutions to the vortex problem, but we do not need the precise form, just the fact that a solution exists. Since Δ carries charge $2e$ we infer that for a solution to have finite energy there must be a magnetic flux with strength $nhc/2e$ associated to the vortex. We will now show that in the presence of vortices with odd strength there are fermionic zero-modes, and that hese zero-modes imply non-Abelian statistics see [56, 57].

The fermionic mean field Hamiltonian (1.124) is quadratic so it can be diagonalized by single-particle operators

$$\phi_{u,v}^\dagger = \int d^2x \left(\psi^\dagger(\mathbf{r})\ \psi(\mathbf{r}) \right) \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} \ , \tag{1.128}$$

and the Heisenberg equation of motion $[\phi_{u,v}^\dagger, H] = E \phi_{u,v}^\dagger$ becomes

$$\underbrace{\begin{pmatrix} H_1 - \mu & \frac{1}{2}\{\Delta, \partial_z\} \\ -\frac{1}{2}\{\Delta^*, \partial_{\bar{z}}\} & -H_1^* + \mu \end{pmatrix}}_{\mathscr{H}} \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} \ . \tag{1.129}$$

The single particle Hamiltonian $\mathscr{H}$ has a number of discrete modes and a continuum. Using the property

$$\sigma_x \mathscr{H}^* \sigma_x = -\mathscr{H} \tag{1.130}$$

we notice that an odd number of discrete modes implies an odd number of zero-modes. Changing the parameters, modes can come down from the continuum and become discrete and vice versa, but because of the property (1.130) they must always come in pairs. Therefore we can with certainty say that one cannot change the parity of the number of zero-modes. If we have one zero-mode we will continue to have (at least) one. If, by chance, there is an extra pair of zero-modes, interactions would most likely gap out two of the three, so for generic parameters we expect exactly one zero mode. Note that (1.130) is not a physical symmetry that can be broken by changing the Hamiltonian. Instead it follows from the structure of the second-quantized Hilbert-space.

The above argument is only valid for a single vortex on the infinite plane. For any finite system there will necessarily be an even number of discrete modes and it seems like the argument fails. However, as long as the vortex modes are exponentially localized, and the vortices are well separated compared to this length scale, the vortices can still be treated as isolated up to small perturbations.

These perturbations will allow tunneling of one vortex zero-mode into another, which turns them into two non-localized non-zero energy modes. But the splitting will be proportional to the tunneling rate, and therefore exponentially small in λ_Z/L, where L is the distance between the vortices, and λ_Z is the size of the vortex zero-modes. We are thus left with proving that the zero-modes are (exponentially) localized, and that there is only one (or an odd number) of them at each vortex. Although we cannot solve the equation (1.129) in general, we can find analytical expressions for the zero-modes if we make some approximations. We begin by considering an isolated vortex and no disorder potential. With this assumption, and taking (r,θ) as polar coordinates centered around the vortex core, the only θ dependence in (1.129) is in the phase of Δ, and we can assume $\Delta = f(r)e^{i\Omega}$ with $\Omega = in\theta + i\lambda$ and λ a constant. Next we notice that the ansatz,

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{il\theta} \begin{pmatrix} e^{i(n-1)\theta/2+i\lambda/2}u_l(r) \\ e^{-i(n-1)\theta/2-i\lambda/2}v_{-l}(r) \end{pmatrix}, \tag{1.131}$$

with $(u_l(r), v_{-l}(r))$ real, diagonalizes $\mathscr{H}$ in the θ variable so we are left with a one-dimensional problem $\mathscr{H}_l(u_l(r), v_{-l}(r))^T = E(u_l(r), v_{-l}(r))^T$. Note that if n is odd l must be an integer, and if n is even, l must be a half integer. Analogous to (1.125) the effective $1d$ Hamiltonian $\mathscr{H}_l$ has the property $\sigma_x H_l^* \sigma_x = -H_{-l}$. So, for each zero energy solution $(u_l(r), v_{-l}(r))^T$ there is necessarily a dual solution, $(v_{-l}(r), u_l(r))^T$. It therefore seems that all zero-modes come in pairs, and we could not possibly get an odd number. However, when the vortex strength n is odd, we have a self-dual solution,

$$(u_0(r), v_0(r))^T = (v_0(r), u_0(r))^T\,,$$

and thus an odd number of zero-modes.

Let us examine the equation for the self-dual mode in more detail. Since $u_0(r) = v_0(r)$ we are left with a single one-dimensional second-order equation. It can be brought to a more familiar form by the gauge choice $\mathbf{A} = A_\theta(r)\hat{\theta}$ and the transformation,

$$u_0(r) = \chi(r)\exp\left(-\frac{m}{2}\int^r f(r')dr'\right)\,.$$

Substituting this into (1.129) gives

$$-\frac{\chi''(r)}{2m} - \frac{\chi'(r)}{2mr} + \frac{((n-1)/2)^2}{2mr^2}\chi(r) + \frac{(n-1)eA_\theta(r)}{2m}\chi(r) + \frac{mf^2}{8}\chi(r) = \mu\chi(r)\,,$$

which we identify as a Schrödinger equation for a particle of mass m, with angular momentum $k = (n-1)/2$, moving in a potential $mf^2/8$, and a radially symmetric magnetic field $B = \frac{1}{r}A_\theta + \partial_r A_\theta$. Further assuming that the coherence length ξ (i.e. the radius of the region where $|\Delta|$ differs substantially from its asymptotic value $|\Delta_0|$) is much smaller than λ_Z, and the London length λ_L (i.e. the radius of region with non-zero magnetic field) is much larger than the same length scale, we obtain the solution,

$$u_0(r) = \begin{cases} e^{-m|\Delta_0|r/2} J_k\left(r\sqrt{2\mu m - (m|\Delta_0|)^2/4}\right) & \text{for } \mu > m|\Delta_0|^2/8 \\ e^{-m|\Delta_0|r/2} I_k\left(r\sqrt{(m|\Delta_0|)^2/4 - 2\mu m}\right) & \text{for } 0 < \mu < m|\Delta_0|^2/8 \,, \end{cases}$$

which is exponentially localized. The approximations we made were to assume that we have an extreme type II superconductor, but the result holds independent of this approximation as long as $\mu > 0$.

We here found the vortex Majorana zero modes from a detailed calculation. However, we want to stress that they are directly related to anyonic nature of the vortices and therefore also the topological order. So, the existence of the modes does not rely on any of the detailed assumptions. As long as the energy gap remains open the zero modes cannot disappear even if the system is changed considerably.

1.7 Fractional Quantum Hall Liquids

Our second example of states with topological interactions are the archetypical ones, namely the quantum Hall liquids. We start with the most celebrated ones.

In 1982, not long after the discovery of the IQHE, Tsui, Stormer and Gossard observed a plateau in the conductance at $\sigma_H = 1/3\sigma_0$ [58]. Later, many more states were discovered with $\sigma_H = p/q\sigma_0$, the vast majority with an odd denominator q. This is the celebrated fractional quantum Hall effect (FQHE). The FQHE poses a much more difficult theoretical challenge than the integer one. The basic difficulty is the massive degeneracy of the free electron states in an partially filled Landau level. Neglecting the lattice potential, the *only* energy scale is that of the Coulomb interaction $E_C \sim e^2\rho^{-1/2}$, where $\rho^{1/2}$ is the mean distance between the particles. (We assume that the cyclotron gap, $E_B \sim eB/m$ is large, so that for all practical purposes $E_C/E_B = 0$.) As a consequence there is no small parameter, and thus no hope to understand the FQHE by using perturbation theory.

The first, and in a sense most successful, approach to the FQH problem was due to [59], who, by an ingenious line of arguments, managed to guess a many-electron wave function, that gives an essentially correct description of the states with conductance $\sigma_H = \sigma_0/m$.

Another approach, which which we will take, is to try to find an effective low energy theory, and we will outline how this can be done. This resulting low-energy theory is called the Chern-Simons-Ginzburg-Landau theory.

1.7.1 The Chern-Simons-Ginzburg-Landau theory

The starting point is the microscopic Hamiltonian for N electrons in a constant transverse magnetic field B, interacting via a two-body potential $V(r)$, which should be thought of as a (suitably screened) Coulomb potential,

$$H = \frac{1}{2m}\sum_{i=1}^{N}(\mathbf{p}_i - eA(\mathbf{x}_i))^2 + \sum_{i<j}^{N} V(|\mathbf{x}_i - \mathbf{x}_j|)\,, \tag{1.132}$$

where $\mathbf{A}(\mathbf{x}) = \frac{B}{2}(-y,x)$. Our aim is to find an equivalent *bosonic* formulation of this theory, which should be amenable to a mean-field description. A direct application of the method of functional bosonization described in Section 1.2.3.1, will not work since we would not be able to compute the partition function $Z[a]$, for a partially filled Landau level. Another approach that might come to mind is to invoke pairing and introduce a Cooper pair field. This will however also not work since it would describe a superconductor, not an insulating QH state. Instead we proceed by first performing a *statistics changing transformation* on the electrons.

The idea is to relate the fermionic wave functions to their bosonic counterparts by the unitary transformation

$$\Psi_F(\mathbf{x}_1,\dots\mathbf{x}_N) = \phi_k(\mathbf{x}_1\dots\mathbf{x}_N)\Psi_B(\mathbf{x}_1,\dots\mathbf{x}_N) \tag{1.133}$$

where the phase factor ϕ_k is given by

$$\phi_k(\mathbf{x}_1\dots\mathbf{x}_N) = e^{ik\sum_{a<b}\alpha_{ab}}\,, \tag{1.134}$$

with k an odd integer, and α_{ab} the polar angle between the vectors $\mathbf{x}_a$ and $\mathbf{x}_b$. The corresponding bosonic Hamiltonian is identical to the fermionic one, except that it includes a coupling to a *statistical, or Chern-Simons, gauge potential*,

$$\mathbf{a}(\mathbf{x}_a) = k\nabla\sum_{b\neq a}\alpha_{ab}\,. \tag{1.135}$$

Thus, instead of studying the original fermionic Hamiltonian (1.132) we can study the bosonic Hamiltonian

$$H_B = \frac{1}{2m}\sum_{a=1}^{N}[\mathbf{p}_a - e\mathbf{A}(\mathbf{x}_a) + \mathbf{a}(\mathbf{x}_a)]^2 - \sum_{a<b} V(|\mathbf{x}_a - \mathbf{x}_b|)\,. \tag{1.136}$$

To proceed we first notice that although $\mathbf{a}$ looks like a pure gauge, it is singular at the positions of the particles, so that the statistical magnetic field is given by,

$$\varepsilon^{ij}\partial_i a_j \equiv b^{(a)} = 2\pi k\sum_{b\neq a}\delta^2(\mathbf{x}_a - \mathbf{x}_b)\,, \tag{1.137}$$

which amounts to *attaching a singular flux tube of strength k to each particle*. The statistical exchange phase, $\theta = k\pi$ can thus be seen as an Aharanov-Bohm effect.[14]

We are now ready to construct a quantum field theory describing our bosonized electron in a path integral formulation. The variables will be a non-relativistic boson field, ϕ, describing the electrons, and the statistical gauge field $\mathbf{a}$. The expression for the statistical magnetic field (1.137) implies that $\mathbf{a}$ is such that there is a flux tube tied to each particle, i.e.,

$$2\pi k\rho = 2\theta\phi^\star\phi = \varepsilon^{ij}\partial_i a_j \,. \tag{1.138}$$

This *local* constraint is implemented by a Lagrange multiplier field a_0, and the result is the non-relativistic Chern-Simons-Ginzburg-Landau (CSGL) field theory,

$$\mathscr{L}_B = \mathscr{L}_\phi + \frac{1}{2\pi k} a_0 \varepsilon^{ij}\partial_i a_j \,, \tag{1.139}$$

where

$$\mathscr{L}_\phi = \phi^\star(i\partial_0 - a_0 + eA_0)\phi - \frac{1}{2m}|(\mathbf{p} + e\mathbf{A} - \mathbf{a})\phi|^2 - V(|\phi|)\,, \tag{1.140}$$

and where A^μ is an external electromagnetic field, that includes the constant background magnetic field B, and $\rho = \phi^\star\phi$ is the density. The term $\sim a_0\varepsilon^{ij}\partial_i a_j$ is nothing but the Coulomb gauge version (i.e., $\nabla\cdot\mathbf{a} = 0$) of the full CS action $\sim \varepsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma$; so, we can finally write the partition function as

$$Z[A_\mu] = \int \mathscr{D}[\phi^\star]\mathscr{D}[\phi]\mathscr{D}[a_\mu]\, e^{i\int d^3x\, \mathscr{L}_{CSGL}(\phi,a;A)} \tag{1.141}$$

with

$$\mathscr{L}_{CSGL} = \mathscr{L}_\phi + \frac{1}{2\pi k}\varepsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma \,. \tag{1.142}$$

To proceed we employ a mean field approach. To do so we first assume V in $\mathscr{L}_\phi$ to be a contact potential,

$$V(|\phi|) = -\frac{\mu}{2}|\phi|^2 + \frac{\lambda}{4}|\phi|^4 \,,$$

where both μ and λ are positive, so the minimum occurs at finite density. In the mean field approach we assume that the thin flux tubes that make up the statistical gauge field $b = \varepsilon^{ij}\partial_i a_j$, can be replaced by a smeared out field that can be cancelled against the constant external field B, i.e., $\varepsilon^{ij}\partial_i a_j = e\varepsilon^{ij}\partial_i A_j$. With this, we can choose a gauge where the combination $eA - \mathbf{a} = 0$, and the vector potential terms in

[14] The naive picture of the "composite bosons" flux-charge composites, is however slightly misleading since that would implies that you get a phase $2\times k2\pi$ when taking one particle a full turn around another; there are equal contributions from the charge circling the flux and the flux circling the charge. This is not what happens, the correct phase is $k2\pi$ corresponding to the exchange phase $k\pi$ [60].

$\mathscr{L}_\phi$ then disappear. The equation of motion for the a_0 field gives the constraint,

$$\varepsilon^{ij}\partial_i a_j = 2\pi k\phi^*\phi = 2\pi k\rho \ , \tag{1.143}$$

which combined with the mean field assumption results in

$$\rho = \phi^*\phi = \frac{eB}{2\pi k} \ .$$

To get the full mean field solution, the mean density $\bar{\rho}$ is picked as to minimise $V(\rho)$, and in summary we have,

$$\phi = \phi_0 = \bar{\rho} = \sqrt{\frac{\mu}{\lambda}} \qquad ; \qquad \mathbf{a} = e\mathbf{A} \ . \tag{1.144}$$

This is only possible if

$$\bar{\rho} = \frac{1}{2\theta}b = \frac{1}{2\pi k}eB = \frac{\rho_0}{k} \tag{1.145}$$

(where ρ_0 is the density of a filled Landau level). Or in other terms, the filling fraction is $\nu = 1/k$, where k can be any odd integer.

Just as in the usual Ginzburg-Landau theory, the GLCS theory supports mean field vortex solutions, which in Coulomb gauge, for a unit strength vortex is,

$$\phi \underset{r\to\infty}{\sim} \sqrt{\bar{\rho}}\, e^{i\varphi} \qquad ; \qquad a_\varphi \underset{r\to\infty}{\sim} \frac{1}{r} \qquad ; \qquad a_r = 0 \ . \tag{1.146}$$

Since the statistical magnetic field is tied to the charge density (1.138), we can calculate the excess charge related to the vortex by integrating the expression for the statistical magnetic field (1.145),

$$Q_v = \int d^2r\,\rho(\mathbf{x}) = \frac{\nu e}{2\pi}\int d^2x\,\varepsilon^{ij}\partial_i a_j = \frac{\nu}{2\pi}e\int d\mathbf{x}\cdot\mathbf{a} = \nu e \ . \tag{1.147}$$

So, the vortex describes a quasi-particle with fractional charge νe. In [61] and [11] you find a more detailed analysis of the CSGL theory including the demonstration that the quasi-particles are Abelian anyons with a statistics angle $\nu\pi$. Here we shall show how it can be used to extract an effective topological theory for the Laughlin states.

1.7.2 From the CSGL theory to the effective topological theory

In the presence of a collection of vortices, we parametrize the field ϕ as

$$\phi = \sqrt{\rho(\mathbf{x})}e^{i\theta(\mathbf{x})}\xi_v(\mathbf{x})\,, \tag{1.148}$$

where we have extracted the singularities in ξ_v and θ is a smooth fluctuating phase. Next we substitute this parametrization (1.148) into the CSLG Lagrangian (1.142) and expand the Lagrangian around the mean-field solution,

$$\phi = \left(\sqrt{\bar{\rho}} + \frac{\delta\rho}{2\sqrt{\bar{\rho}}} \right) \xi_v e^{i\theta} \,, \tag{1.149}$$

and the temporal term then become

$$\mathscr{L}_{temp} = \delta\rho \left(-\partial_t \theta + i\xi_v^\star \partial_t \xi_v - a_0 + eA_0 \right) + \mathscr{O}\left(\delta\rho^2\right) + \mathscr{O}\left(\delta\rho(eA_0 - a_0)\right) \,. \tag{1.150}$$

The term $\bar{\rho}\partial_t\delta\rho$ vanishes when the derivative is integrated over and does not contribute. The leading contribution from the kinetic energy is

$$\mathscr{L}_{kin} = -\frac{\bar{\rho}^2 + \mathscr{O}(\delta\rho)}{2m} \left(\nabla\theta - i\xi_v^* \nabla \xi_v - \mathbf{a} + e\mathbf{A} \right)^2 + \mathscr{O}\left(\partial_i \delta\rho\right) \,, \tag{1.151}$$

and we linearize this quadratic term by introducing a Hubbard Stratonovich field $\mathbf{X}$, to get,

$$\mathscr{L}_{kin} = \frac{m}{2\bar{\rho}} \mathbf{X}^2 + \mathbf{X} \cdot \left(\nabla\theta - i\xi_v^* \nabla \xi_v - \mathbf{a} + e\mathbf{A} \right) + h.o. \,, \tag{1.152}$$

where $h.o.$ denote terms of higher order in the expansion. We now recognise θ as a Lagrange multiplier field, imposing conservation of the three-current $(\delta\rho, \mathbf{X})$, which therefore can be parametrized in terms of a field b_μ as

$$(\delta\rho, \mathbf{X})^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\sigma} \partial_\nu b_\sigma \,. \tag{1.153}$$

The terms containing ξ_v can now be taken together and written as

$$\frac{1}{2\pi i} \varepsilon^{\mu\nu\sigma} \left(\xi_v^* \partial_\mu \xi_v \right) \partial_\nu b_\sigma \,. \tag{1.154}$$

To interpret this term, let's look closer at ξ_v. The integral

$$\int_\gamma d\mathbf{l} \cdot \xi_v^* \nabla \xi_v \tag{1.155}$$

gives the change of the phase of ξ_v along the curve γ and thus, by definition, if γ is a closed curve

$$\frac{1}{2\pi i} \int_\gamma d\mathbf{l} \cdot \xi_v^* \nabla \xi_v = q_v \,, \tag{1.156}$$

where q_v denotes the number of right handed minus the number of left handed vortices encircled by γ, i.e., the vortex charge encircled by γ. Using the $2d$ version of Stokes theorem we can thus conclude

$$\frac{1}{2\pi i} \int_S \varepsilon^{ij} \partial_i \xi_v^* \partial_j \xi_v = q_v \,, \tag{1.157}$$

where q_v is the vortex charge in the region S. It follows that the vortex charge density is

$$\rho_v = \frac{1}{2\pi i}\varepsilon^{ij}\partial_i\xi_v^*\partial_j\xi_v \ . \tag{1.158}$$

Generalizing this argument by letting γ be a general space-time curve, we can write the full vortex current as

$$j_v^\mu = \frac{1}{2\pi i}\varepsilon^{\mu\nu\sigma}\partial_\nu\xi_v^*\partial_\sigma\xi_v \ . \tag{1.159}$$

By integration by parts, of the derivative ∂_ν, and using the above identity for j_v^μ, (1.159), the term that contain the ξ_v of the Lagrangian (1.154) can be written as $b_\mu j_v^\mu$. Collecting everything, we get the full Lagrangian,

$$\mathscr{L} = \frac{1}{2\pi}\varepsilon^{\mu\nu\sigma}(eA_\mu - a_\mu)\partial_\nu b_\sigma + b_\mu j_v^\mu + \frac{1}{4\pi k}\varepsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma + \ h.o \ , \tag{1.160}$$

where *h.o.* denote all higher derivative terms in b_μ. Finally, integrating out a_μ we end up with

$$\mathscr{L}_{top} = -\frac{k}{4\pi}\varepsilon^{\mu\nu\sigma}b_\mu\partial_\nu b_\sigma - \frac{e}{2\pi}\varepsilon^{\mu\nu\sigma}A_\mu\partial_\nu b_\sigma + b_\mu j_v^\mu \ . \tag{1.161}$$

Note that for a filled Landau level, i.e. $\nu = 1$, this exactly reproduces the previously derived expression for the topological Lagrangian (1.27), and just as before the gauge field b parametrizes the current. For the Laughlin states $k = 3, 5, \ldots$ the theory superficially looks very similar, but the higher integer values of k have far reaching consequences: fractional charge and statistics for the quasi-particles, ground-state degeneracy on higher genus surfaces, and chiral bosonic edge modes with k-dependent correlation functions [33].

1.7.3 The Abelian hierarchy

In experiments on very clean samples, one sees a large number of FQH states [62]. Most of them fit beautifully into a hierarchical scheme [63, 64], where "daughter" states are formed by the condensation of anyonic quasi-particles in a "parent" FQH state, just as the Laughlin states can be thought of as a condensation of the original electrons.

Starting from the just derived topological Lagrangian (1.161) we can deduce what possible topological states can emerge from condensing quasi-holes. To do so, we need a dynamical theory for the holes, that in principle could be obtained by keeping higher order terms in above derivation of the effective theory. This is however difficult so we shall use a heuristic approach.

The basic, and quite reasonable, assumption is that the condensation of the quasi-holes can be described by the same procedure as was used above for the electrons.

This amounts to taking the following Ginzburg-Landau theory for the quasiholes,

$$\mathscr{L}_{\xi_v} = \phi_v^\star(i\partial_0 - b_0)\phi_v - \frac{1}{2m}|(\mathbf{p}+\mathbf{b})\phi_v|^2 - V(\rho_v)\,, \tag{1.162}$$

where ϕ_v is a bosonic vortex field associated to the quasiholes. We can again introduce a statistical gauge potential a and couple it to ϕ_v, but this time with the coefficient $1/4\pi k_2$ where k_2 is an even integer,

$$\tilde{\mathscr{L}}_{\xi_v} = \phi_v^\star(i\partial_0 + b_0 - a_0)\phi_v - \frac{1}{2m}|(\mathbf{p}+\mathbf{b}-\mathbf{a})\phi_v|^2 - V(\rho_v) + \frac{1}{4\pi k_2}\varepsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma\,. \tag{1.163}$$

This is a trivial statistical transmutation, which adds the phase $e^{i2\pi k}$ to each particle exchange. In the mean-field approach, it will however have the effect to make the hole condensate thinner, just as in the original electron condensation. We can now, as before, find a mean-field solution where a_μ cancel against b_μ. The equation of motion for the density is obtained by varying b_0 and a_0,

$$\rho_v = -\frac{k}{2\pi}\varepsilon^{ij}\partial_i b_j + \frac{e}{2\pi}B \qquad ; \qquad \rho_v = \frac{1}{2\pi k_2}\varepsilon^{ij}\partial_i a_j\,. \tag{1.164}$$

Combining this with the requirement $\varepsilon^{ij}\partial_i b_j = \varepsilon^{ij}\partial_i a_j$ imply that the mean-field solution is possible only when $\rho_c = \varepsilon^{ij}\partial_i b_j/2\pi$ equals

$$\rho_c = \frac{1}{2\pi}\frac{eB}{k+\frac{1}{k_2}} = \rho_0\frac{1}{k+\frac{1}{k_2}}\,.$$

To obtain the vortices in the field ϕ_v we can do almost exactly as above, with the result,

$$\mathscr{L}_{top} = -\frac{k}{4\pi}\varepsilon^{\mu\nu\sigma}b_\mu\partial_\nu b_\sigma - \frac{k_2}{4\pi}\varepsilon^{\mu\nu\sigma}b^2_\mu\partial_\nu b^2_\sigma - \frac{1}{4\pi}\varepsilon^{\mu\nu\sigma}b_\mu\partial_\nu b^2_\sigma - \frac{1}{4\pi}\varepsilon^{\mu\nu\sigma}b^2_\mu\partial_\nu b_\sigma - \frac{e}{2\pi}\varepsilon^{\mu\nu\sigma}A_\mu\partial_\nu b_\sigma + b_\mu j^\mu_{v1} + b^2_\mu j^\mu_{v2}\,.$$

To see how this generates a whole hierarchy of states, we consider condensation of the vortices in the previous state and so on. At level n that is with n consecutive condensations we end up with n gauge fields and the Lagrangian

$$\mathscr{L} = -\frac{1}{4\pi}\sum_{\alpha,\beta}K_{\alpha\beta}\varepsilon^{\mu\nu\sigma}b^\alpha_\mu\partial_\nu b^\beta_\sigma - \sum_\alpha \frac{e}{2\pi}t_\alpha A_\mu\varepsilon^{\mu\nu\sigma}\partial_\nu b^\alpha_\sigma + \sum_\alpha l_\alpha b^\alpha_\mu j^\mu\,, \tag{1.165}$$

where the *K-matrix* K and the the *charge vector* t takes the values

$$K_{\alpha\beta} = k_\alpha\delta_{\alpha\beta} - \delta_{\alpha+1,\beta} - \delta_{\alpha,\beta+1} \qquad ; \qquad t_\alpha = \delta_{\alpha,1}\,, \tag{1.166}$$

and where the vector **l** is integer valued and describes all possible the quasi-particles excitations. As we realize from how this was constructed k_α is an even integer except for k_1 which is odd. It is explained in detail in [17] how to extract the topological information from this Lagrangian, and here we just quote the following important results for the filling fraction ν, and the charge (Q_α) and statistics angle (θ_α) for the α^{th} quasi-particle:

$$\nu = \mathbf{t}^T \mathbf{K}^{-1} \mathbf{t} \qquad ; \qquad Q_\alpha = -e \mathbf{t}^T \mathbf{K}^{-1} \mathbf{l} \qquad ; \qquad \theta_\alpha = \pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l} . \tag{1.167}$$

1.8 Mathematical background and proofs

In this section you will find mathematical background that complement the main text of this chapter. You will also find proofs, too technical for the main text, of some of the statements made in the text.

1.8.1 Vector bundles and Chern numbers in quantum mechanics

Consider a N dimensional subspace $h(\mathbf{b})$ of a Hilbert space $\mathscr{H}$ that varies continuously with the parameters $\mathbf{b}$ in some manifold $\mathbf{b} \in B$. The space

$$E = \bigcup_{\mathbf{b} \in B} \{\mathbf{b}\} \times h(\mathbf{b}) \,,$$

which is a subspace of $B \times \mathscr{H}$, is an example of a fiber bundle. The manifold B is called the base space, and the space $h(\mathbf{b})$ is called the fiber at $\mathbf{b}$. More specifically this is an example of a complex vector bundle, since the fibers are complex vector spaces. In the main text we encounter two distinct cases:

1. A quantum mechanical many-body system on a $2d$ torus with fluxes $\phi = (\phi_x, \phi_y)$ through the two holes. For each value the fluxes there are N degenerate ground-states (recall figure 1.2 in section 1.2.1). In this case the base manifold $B \equiv T^2_\phi$ is the flux-torus from section 1.2.1, $\mathscr{H}$ is the many-body Hilbert space and $h(\phi)$ is the N-dimensional sub-space consisting of the degenerate ground-states at ϕ.
2. A systems of non-interacting fermions on a lattice. The base manifold $B \equiv B.Z.$ is the Brillouin-zone torus, $\mathscr{H}$ is the single-particle Hilbert space and $h(\mathbf{k})$ is the space of occupied single-particle states at lattice momentum $\mathbf{k}$.

Complex fiber-bundles are geometrical structures which have certain characteristics which take discrete vales, and which therefore cannot be altered by continuous transformations. Below we will construct one set of such characteristics, the Chern numbers. For the two above examples of fiber bundles, these Chern numbers are related to experimentally measurable transport coefficients.

We will also describe another set of characteristics called the Chern-Simons invariants. Contrary to the Chern numbers they are not topological invariants of the fiber-bundle, but for the two-mentioned examples, they are quantized if certain symmetries are invoked.

Both these quantities are defined in terms of a Berry connection which we will now define.

1.8.1.1 The Berry connection

The Berry connection $\mathscr{A}_\mu$ is an operator taking $h(\mathbf{b})$ to $h(\mathbf{b}+db^\mu)$; it relates an element $|\mathbf{b}\rangle \in h(\mathbf{b})$ to the element in $h(\mathbf{b}+db^\mu)$ closest to $|\mathbf{b}\rangle$. If $h(\mathbf{b})$ is independent of $\mathbf{b}$ the closest element to $|\mathbf{b}\rangle$ in $h(\mathbf{b}+db^\mu)$ trivially is $|\mathbf{b}\rangle$ itself, while if $h(\mathbf{b})$ varies with $\mathbf{b}$ that is not necesarrily the case. The notion of distance in the Hilbert space $\mathscr{H}$ is given by the inner product and the vector closest to $|\mathbf{b}\rangle$ in $h(\mathbf{b}+db^\mu)$ is the orthogonal projection of $|\mathbf{b}\rangle$ into $h(\mathbf{b}+db^\mu)$,

$$P\left(\mathbf{b}+db^\mu\right)|\mathbf{b}\rangle \ . \tag{1.168}$$

The Berry connection is defined by

$$P\left(\mathbf{b}+db^\mu\right)|\mathbf{b}\rangle = \left(\mathbb{1} - i\mathscr{A}_\mu db^\mu\right)|\mathbf{b}\rangle \ , \tag{1.169}$$

where $\mathscr{A}_\mu$ are operators from $h(\mathbf{b})$ to $h(\mathbf{b}+db^\mu)$.

Although this abstract form sometimes is useful we usually have to choose a specific basis to do any calculation. We therefore pick a basis

$$\{|\mathbf{b};\alpha\rangle\}_{\alpha=1,\dots,N} \ , \tag{1.170}$$

that varies smoothly in some region B and we can write

$$|\mathbf{b}\rangle = \sum_\alpha a_\alpha(\mathbf{b})\,|\mathbf{b};\alpha\rangle \ , \tag{1.171}$$

for some coefficients $a_\alpha(\mathbf{b})$. In this basis the projection operator from $h(\mathbf{b})$ to $h(\mathbf{b}+db^\mu)$ is represented by,

$$\langle \mathbf{b}+db^\mu;\alpha|P\left(\mathbf{b}+db^\mu\right)|\mathbf{b};\beta\rangle = \delta_\alpha^\beta - \langle\mathbf{b};\alpha|\frac{\partial}{\partial b^\mu}|\mathbf{b};\beta\rangle\, db^\mu \ , \tag{1.172}$$

(the relation $\left(\frac{\partial}{\partial b^\mu}\langle\mathbf{b};\alpha|)|\mathbf{b};\beta\rangle\right) = -\langle\mathbf{b};\alpha|\frac{\partial}{\partial b^\mu}|\mathbf{b};\beta\rangle$ was used) and we get the expression

$$a_\alpha(\mathbf{b}) - \sum_\beta \langle\mathbf{b};\alpha|\frac{\partial}{\partial b^\mu}|\mathbf{b};\beta\rangle\, a_\beta(\mathbf{b})db^\mu \tag{1.173}$$

for the left hand side of the definition (1.169) of the Berry connection. We can then read of

$$\mathscr{A}_{\alpha\mu}^{\ \beta} = i\,\langle\mathbf{b};\alpha|\frac{\partial}{\partial b^\mu}|\mathbf{b};\beta\rangle \ . \tag{1.174}$$

1.8.1.2 The Berry field strength

The Berry connection relates coefficients of vectors in two different Hilbert spaces, one at $h(\mathbf{b})$ and the other at $h(\mathbf{b}+db^\mu)$. Thus, the matrix $\mathscr{A}_{\alpha\mu}^{\beta}$ can be taken arbitrary since one could change the basis of $h(\mathbf{b})$ independent from the basis of $h(\mathbf{b}+db^\mu)$. The representation, $\mathscr{A}_{\alpha\mu}^{\beta}$, thus cannot be used to characterise the fiber bundle, we clearly have to look for something else.

The strategy is to consider a closed curve to define a matrix acting in a single Hilbert space $h(\mathbf{b})$. A matrix is a representation of an operator, and the precise form of the matrix is *not* basis independent, the trace however is, and we now show how it can be used to define the Chern numbers.

Consider an infinitesimal loop obtained by first moving the db^μ in the b^μ-direction, then db^ν in the ν-direction, db^μ backward in the μ-direction and finally back to where we started. We start out with the fiber $|\mathbf{b}\rangle$ then take the fiber closest to it in $h(\mathbf{b}+db^\mu)$, i.e.,

$$P(\mathbf{b}+db^\mu)\,|\mathbf{b}\rangle \ , \tag{1.175}$$

then take the fiber closest to that in $h(\mathbf{b}+db^\mu+db^\nu)$ etc. until the loop is closed; we end up with

$$P(\mathbf{b})\,P(\mathbf{b}+db^\nu)\,P(\mathbf{b}+db^\mu+db^\nu)\,P(\mathbf{b}+db^\mu)\,|\mathbf{b}\rangle \ . \tag{1.176}$$

The Berry field strength $\mathscr{F}_{\mu\nu}$ is defined from this expression by

$$P(\mathbf{b})\,P(\mathbf{b}+db^\nu)\,P(\mathbf{b}+db^\mu+db^\nu)\,P(\mathbf{b}+db^\mu)\,|\mathbf{b}\rangle = \left(\mathbb{1} - i\mathscr{F}_{\mu\nu}db^\mu db^\nu\right)|\mathbf{b}\rangle \ . \tag{1.177}$$

By Taylor expanding the projectors it is straight forward to write the field strength in terms on the connection,

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu + i\mathscr{A}_\mu \mathscr{A}_\nu - i\mathscr{A}_\nu \mathscr{A}_\mu \ , \tag{1.178}$$

or written as matrices in a specific basis,

$$\mathscr{F}_{\alpha\mu\nu}^{\ \beta} = \partial_\mu \mathscr{A}_{\alpha\nu}^{\beta} - \partial_\mu \mathscr{A}_{\alpha\nu}^{\beta} + i\mathscr{A}_{\alpha\mu}^{\gamma}\mathscr{A}_{\gamma\nu}^{\beta} - i\mathscr{A}_{\alpha\nu}^{\gamma}\mathscr{A}_{\gamma\mu}^{\beta} \ , \tag{1.179}$$

where summation over repeated indices is understood. If B is two-dimensional $\mathscr{F}_{\mu\nu}$ has only one non-trivial component and we suppress the lower indices and use the notation

$$\mathscr{F} \equiv \mathscr{F}_{12} = -\mathscr{F}_{21} = \frac{1}{2}\varepsilon^{\mu\nu}\mathscr{F}_{\mu\nu} \ . \tag{1.180}$$

The field strenght $\mathscr{F}_{\mu\nu}$ is an operator that act within a fiber $h(\mathbf{b})$ and the trace of it equals the sum of the diagonal components

$$\mathrm{Tr}[\mathscr{F}_{\mu\nu}] = \mathscr{F}_{\alpha\mu\nu}^{\ \alpha} \ , \tag{1.181}$$

in any basis (rememeber that the repeated index α is summed over). The Berry connection on the other hand is different. $\mathscr{A}_\mu(\mathbf{b})$ is an operator from the fiber $h(\mathbf{b})$ to the fiber $h(\mathbf{b}+b^\mu)$ and an operator between two different spaces have no notion of a trace.[15] We will still write $\mathrm{Tr}[\mathscr{A}_\mu]$ (and similar for products $\mathrm{Tr}[\mathscr{A}_\mu\mathscr{A}_\nu]$ etc.) as a short-hand for $\mathscr{A}_{\alpha\mu}^{\ \ \alpha}$, but you have to remember that this *is a basis dependent expression*. It is instructive to see how this comes about in an explicit calculation: From the definition of the Berry connection (1.169) it follows, that under a coordinate transformation,

$$|\mathbf{b};\alpha\rangle \to \sum_\beta U_{\alpha\beta}(\mathbf{b})\,|\mathbf{b};\beta\rangle \ ,$$

the coefficients of the Berry connection transforms as

$$\mathscr{A}_{\alpha\mu}^{\ \ \beta} \to U_\alpha^{\ \gamma}\mathscr{A}_{\gamma\mu}^{\ \ \delta}U_\delta^{\dagger\ \beta} - iU_\alpha^{\dagger\ \gamma}\partial_\mu U_\gamma^{\ \beta} \ , \tag{1.182}$$

while the field strength just rotates,

$$\mathscr{F}_{\alpha\mu\nu}^{\ \ \ \beta} \to U_\alpha^{\ \gamma}\mathscr{F}_{\gamma\mu\nu}^{\ \ \ \delta}U_\delta^{\dagger\ \beta} \ . \tag{1.183}$$

So from the cyclic property one realize that $\mathscr{F}_{\alpha\mu\nu}^{\ \ \ \alpha}$ is basis independent and equals the trace of the operator $\mathscr{F}_{\mu\nu}$. One the other hand, $\mathrm{Tr}[\mathscr{A}_\mu] \equiv \mathscr{A}_{\alpha\mu}^{\ \ \alpha}$ transforms as

$$\mathrm{Tr}[\mathscr{A}_\mu] \to \mathrm{Tr}[\mathscr{A}_\mu] - iU_\alpha^{\dagger\ \gamma}\partial_\mu U_\gamma^{\ \alpha} \tag{1.184}$$

under a basis transformation.

We will end this section with a useful formula for the trace of $\mathscr{F}_{\mu\nu}$. Note that in the expression for the Berry field-strength (1.179) the two terms without derivatives come with different sign and thus vanishes when traced over, because of the cyclic property of the trace,

$$\mathrm{Tr}\left[-i\mathscr{A}_\mu\mathscr{A}_\nu + i\mathscr{A}_\nu\mathscr{A}_\mu\right] \equiv i\mathscr{A}_{\alpha\mu}^{\ \ \gamma}\mathscr{A}_{\gamma\nu}^{\ \ \alpha} - i\mathscr{A}_{\alpha\nu}^{\ \ \gamma}\mathscr{A}_{\gamma\mu}^{\ \ \alpha} = 0 \ . \tag{1.185}$$

One can therefore write

$$\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\right) \equiv \mathscr{F}_{\alpha\mu_1\nu_1}^{\ \ \ \ \ \alpha} = -i\sum_\alpha\left(\frac{\partial}{\partial b^\mu}\left\langle \mathbf{b};\alpha\left|\frac{\partial}{\partial b^\nu}\right|\mathbf{b};\alpha\right\rangle - \frac{\partial}{\partial b^\nu}\left\langle \mathbf{b};\alpha\left|\frac{\partial}{\partial b^\mu}\right|\mathbf{b};\alpha\right\rangle\right) \ . \tag{1.186}$$

[15] You might object since $\mathscr{A}_\mu(\mathbf{b})$ is an operator within the full Hilbert space $\mathscr{H}$, so one should be able to define its trace. That is, in principle, a correct assumption but the trace of $\mathscr{A}_\mu(\mathbf{b})$ does not depend only on the structure of the fibers but of the full Hilbert space $\mathscr{H}$ and it does not necessarily equal $\mathscr{A}_{\alpha\mu}^{\ \ \alpha}(\mathbf{b})$, in a particuliar basis. The reason is the fact that the basis $\{|\mathbf{b};\alpha\rangle\}$ varies with $\mathbf{b}$; $\mathscr{A}_{\alpha\mu}^{\ \ \beta}(\mathbf{b})$ is the represnetation of the operator $\mathscr{A}_\mu(\mathbf{b})$ where the bras are written in the basis $\{|\mathbf{b}+b^\mu;\alpha\rangle\}$ and the kets in the basis $\{|\mathbf{b};\alpha\rangle$, see (1.172).

1.8.1.3 First Chern number and Chern-Simons invariant

Let us now assume that B is two-dimensional (or think of a two-dimensional submanifold of a general B), and form,

$$ch_1[\mathscr{F}](S) = \frac{1}{4\pi}\int_B d^2b\,\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{F}_{\mu\nu}]\ . \tag{1.187}$$

This expression defines the first Chern number, ch_1, and the integrand (including the prefactor) is called the first Chern character.

One can in general cannot define a basis for each $h(b)$ that varies continuously for all $b \in B$. For any region topologically equivalent to a subset of $\mathbb{R}^n$ one can define a continuous basis. If B is topologically equivalent to a disc we can define a continuous basis in the whole of B, and we can use the basis dependent expression $\mathscr{A}_{\alpha\mu}^{\beta}$ throughout B. Then by using $\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{A}_\mu\mathscr{A}_\nu] = 0$ (remember that $\mathrm{Tr}[\mathscr{A}_\mu\mathscr{A}_\nu]$ is defined by the basis dependent expression $\mathscr{A}_{\alpha\mu}^{\beta}$) we can write $\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{F}_{\mu\nu}] = 2\varepsilon^{\mu\nu}\partial_\mu\mathrm{Tr}[\mathscr{A}_\nu]$. This allows us to use Stokes theorem to rewrite the Chern number as

$$\frac{1}{2\pi}\int_{\partial B} dl^\mu\,\mathrm{Tr}[\mathscr{A}_\mu]\,. \tag{1.188}$$

The integrand (again including the prefactor) is called the first Chern-Simons form. Next we assume that B is a sphere[16] and to evaluate the integral in (1.187) we cover the sphere with two caps, the northern hemisphere (α) and the southern hemisphere (β). With this we have

$$\frac{1}{4\pi}\int_B d^2b\,\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{F}_{\mu\nu}] = \frac{1}{4\pi}\int_\alpha d^2b\,\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{F}_{\mu\nu}] + \frac{1}{4\pi}\int_\beta d^2b\,\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{F}_{\mu\nu}]\ . \tag{1.189}$$

Since both the caps are topologically equivalent to the disc, we can in each one of them pick a continuous basis. Using Stokes theorem we get,

$$\frac{1}{4\pi}\int_B d^2b\,\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{F}_{\mu\nu}] = \frac{1}{2\pi}\int_{\partial\alpha} dl^\mu\,\mathrm{Tr}[\mathscr{A}_\mu^{(\alpha)}] - \frac{1}{2\pi}\int_{\partial\alpha} dl^\mu\,\mathrm{Tr}[\mathscr{A}_\mu^{(\beta)}]\ , \tag{1.190}$$

where we used the fact that $\partial\beta$ is the same curve as $\partial\alpha$, but with opposite orientation, and where the superscript α means that $\mathscr{A}_\mu^{(\alpha)}$ is defined with respect to a coordinate system that is continous in the region α, and similarly for $\mathscr{A}_\mu^{(\beta)}$. On the equator both coordinate systems are continuous and they are thus related by some unitary transformation U,

$$\mathrm{Tr}[\mathscr{A}_\mu^{(\beta)}] = \mathrm{Tr}[\mathscr{A}_\mu^{(\alpha)}] - i\mathrm{Tr}[U^\dagger\partial_\mu U]\ . \tag{1.191}$$

[16] The fact that the first Chern number is an integer, proven here, hold for a general closed manifold as e.g. the torus. The proof is more involved since one has to divide B into more regions, but the arguments are analogous to the one for the sphere.

Putting this together we get

$$\frac{1}{4\pi}\int_B d^2b\varepsilon^{\mu\nu}\mathrm{Tr}[\mathscr{F}_{\mu\nu}] = \frac{1}{2\pi i}\int_{\partial\alpha} dl^\mu \mathrm{Tr}[U^\dagger\partial_\mu U] \ . \tag{1.192}$$

The matrix trace is the same in all basis so we can consider it in the basis where U is diagonal

$$U = \mathrm{diag}\left(e^{i\theta_1}, e^{i\theta_2}, \dots\right) \ , \tag{1.193}$$

to get

$$ch_1(S) = \frac{1}{2\pi i}\int_\gamma dl^\mu \, \mathrm{Tr}[U^\dagger\partial_\mu U] = \frac{1}{2\pi i}\sum_i \int_{\partial\alpha} dl^\mu e^{-i\theta_i}\partial_\mu e^{i\theta_i} \ . \tag{1.194}$$

The i^{th} term in this sum gives the change of the phase angle θ_i accumulated when integrating over $\partial\alpha$, but since this is a closed curve this change has to equal $2\pi n$ for some integer n, which completes the proof that $ch_1(S)$ is an integer.

The formula (1.191) allows for another important conclusion. Since $\mathrm{Tr}[\mathscr{A}_\mu]$ changes by $-i\mathrm{Tr}[U^\dagger\partial_\mu U]$ under a basis transformation and $\int dl^\mu \mathrm{Tr}[U^\dagger\partial_\mu U] = 2\pi n i$ for some integer n, the exponent of 2π times the first Chern-Simons invariant

$$CS_1[\mathscr{A}](\gamma) = \frac{1}{2\pi}\int_\gamma dl^\mu \mathrm{Tr}[\mathscr{A}_\mu]$$

is also a basis independent quantity, meaning that $\exp(iCS_1)$ is well defined. It is however not necessarily quantized as an integer.

1.8.1.4 Relating the Brillouin-zone and the flux-torus Chern numbers

For a non-interacting fermionic system on lattice with periodic boundary conditions we can define two different fiber bundles. The first is the flux-torus fiber bundle with the flux torus T^2_ϕ as base space and fibers $h(\phi)$ the one-dimensional Hilbert spaces spanned by $|GS;\phi\rangle$ (i.e., the ground-state at flux ϕ which are the fluxes encircled by the two independent non-contractible loops on the torus).

The other fiber bundle has the Brillouin-zone as the base space and the fibers at lattice momentum $\mathbf{k}$ are spanned by the single particle wave functions,

$$\psi_{\mathbf{k}n}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}u_{\mathbf{k}n}(\mathbf{x}) \ , \tag{1.195}$$

of the the N filled bands. The Berry connection over the Brillouin-zone fiber bundle is given by

$$\mathscr{A}^{\ m}_{nk_i}(\mathbf{k}) = -i\,\langle\psi_{\mathbf{k}n}(\mathbf{x})|\partial_{k_i}|\psi_{\mathbf{k}n}(\mathbf{x})\rangle \equiv \int d^2x\,\psi^*_{\mathbf{k}n}(\mathbf{x})\partial_{k_i}\psi_{\mathbf{k}n}(\mathbf{x}) \ . \tag{1.196}$$

This can also be written as the anti-commutator of the creation and annihilation operators,

$$\mathscr{A}_{n k_i}^{\ m}(\mathbf{k}) = -i\left\{a_{\mathbf{k}n}, \partial_{k_i} a^\dagger_{\mathbf{k}m}\right\} \tag{1.197}$$

where

$$a^\dagger_{\mathbf{k}n} = \int d^2x\, \psi^\dagger(\mathbf{x})\psi_{\mathbf{k}n}(\mathbf{x}) \qquad ; \qquad a_{\mathbf{k}n} = \int d^2x\, \psi(\mathbf{x})\psi^*_{\mathbf{k}n}(\mathbf{x})) \tag{1.198}$$

and $\psi^\dagger(\mathbf{x})$ is the operator that creates an electron at position $\mathbf{x}$,

$$\left\{\psi^\dagger(\mathbf{x}), \psi(\mathbf{x}')\right\} = \delta^2(\mathbf{x}-\mathbf{x}')\ . \tag{1.199}$$

The corresponding Berry field-strength is (which we denote by $\mathscr{B}$ to distinguish it from the previously defined flux torus field strength $\mathscr{F}$)

$$\mathscr{B}_{n k_i k_j}^{\ m} = \partial_{k_i}\mathscr{A}_{n k_j}^{\ m} - \partial_{k_j}\mathscr{A}_{n k_i}^{\ m} + i\mathscr{A}_{n k_i}^{\ p}\mathscr{A}_{p k_j}^{\ m} - i\mathscr{A}_{n k_j}^{\ p}\mathscr{A}_{p k_i}^{\ m}\ , \tag{1.200}$$

where the repeated index p should be summed over. Since the Brillouin-zone is two-dimensional the field-strength has only one independent component

$$\mathscr{B} \equiv \varepsilon^{ij}\mathscr{B}_{k_i k_j}\ . \tag{1.201}$$

We can then write

$$\mathrm{Tr}[\mathscr{B}] = \varepsilon^{ij}\,\mathrm{Tr}[\partial_{k_i}\mathscr{A}_{k_j}] = -i\sum_n \left\{a_{\mathbf{k}n}, \partial_{k_i} a^\dagger_{\mathbf{k}m}\right\}\ , \tag{1.202}$$

for the the defining component of the Brillouin-zone Berry field-strength. Now turn to the other fiber bundle. The fibers are spanned by the ground-states at flux ϕ,

$$|GS;\phi\rangle = \prod_n \prod_{\mathbf{k}\in B.Z.} a^\dagger_{\mathbf{k}n\phi}\,|0\rangle\ , \tag{1.203}$$

where

$$a^\dagger_{\mathbf{k}n\phi} = \int d^2x\, \psi^\dagger(\mathbf{x})\psi_{\mathbf{k}n\phi}(\mathbf{x})\ , \tag{1.204}$$

and $\psi_{\mathbf{k}n\phi}(\mathbf{x})$ is the single particle wave function in band n, at lattice momentum $\mathbf{k}$ and at flux ϕ. The Berry connection is given by

$$\mathscr{A}_{\phi_i} = -i\,\langle GS;\phi|\partial_{\phi_i}|GS;\phi\rangle \tag{1.205}$$

and the defining component of the Berry field strength is

$$\mathscr{F} = -i\varepsilon^{ij}\partial_{\phi_i}\left\langle GS\right|\partial_{\phi_j}\left|GS\right\rangle\ . \tag{1.206}$$

Using the commutation realtions

$$\left\{a_{\mathbf{k}m\phi},a^{\dagger}_{\mathbf{k}'n\phi}\right\} = \delta_{mn}\delta_{\mathbf{k}\mathbf{k}'} \quad ; \quad \left\{a_{\mathbf{k}m\phi},a_{\mathbf{k}'n\phi}\right\} = \left\{a_{\mathbf{k}m\phi},a^{\dagger}_{\mathbf{k}'n\phi}\right\} \quad = 0\ , \tag{1.207}$$

we can write

$$\langle GS;\phi|\partial_{\phi_i}|GS;\phi\rangle = \sum_n \int d^2k \left\{a_{\mathbf{k}n\phi}, \partial_{\phi_i} a^{\dagger}_{\mathbf{k}n\phi}\right\}\ , \tag{1.208}$$

and

$$\mathscr{F}(\phi_i,\phi_j) = -i\varepsilon^{ij}\partial_{\phi_i}\sum_n \int d^2k \left\{a_{\mathbf{k}n\phi}, \partial_{\phi_j} a^{\dagger}_{\mathbf{k}n\phi}\right\}\ . \tag{1.209}$$

We now pick the gauge potential

$$\mathbf{A} = \tilde{\mathbf{A}} + \frac{\phi_x}{L_x}\hat{x} + \frac{\phi_y}{L_y}\hat{y}, \tag{1.210}$$

where the integral of $\tilde{A}_i$ along any of the non-contractible loops on the torus is zero. With this, the Bloch Hamiltonian become

$$H_{Bl} = \frac{\hbar^2}{2m}\left(-i\nabla + e\tilde{\mathbf{A}} + \mathbf{k} + \frac{2\pi}{\phi_0}\left(\phi_x/L_x, \phi_y/L_y\right)\right)^2 + V_{lat}\ . \tag{1.211}$$

Comparing with the Bloch Hamiltonian at zero flux,

$$H_{Bl} = \frac{\hbar^2}{2m}\left(-i\nabla + e\mathbf{A} + \mathbf{k}\right)^2 + V_{lat}\ , \tag{1.212}$$

we realize that we can put $\phi = 0$ and replace

$$\partial_{\phi_i} \to \frac{2\pi}{L_i}\frac{1}{\phi_0}\partial_{k_i}\ , \tag{1.213}$$

which results in

$$\mathscr{F}(\phi) = -i\frac{\varepsilon^{ij}}{\phi_0^2}\int d^2k\ \partial_{k_i}\left\{a_{\mathbf{k}n}, \partial_{k_j}a^{\dagger}_{\mathbf{k}n}\right\} = \frac{1}{\phi_0^2}\int d^2k\ \mathrm{Tr}\left(\mathscr{B}\right)\ , \tag{1.214}$$

where we used (1.202). Finally integrating over the flux torus concludes the proof that the two Chern numbers are equal.

1.8.2 How to normalize the current

Here we shall study the first term $\sim \varepsilon^{\mu\nu\sigma}b_\mu\partial_\nu a_\sigma$ in (1.42) a bit more carefully. Just as the Chern-Simons term this is a topological action, and the two fields a and b have no bulk dynamics. This is however not the full story. If the system is defined on a finite area, $L_x \times L_y$, with periodic boundary conditions, the zero-modes of the fields do acquire dynamics. To understand this, we write the action on the Hamiltonian form,

$$
\begin{aligned}
S_{BF} &= \frac{k}{2\pi}\int d^3x\,\varepsilon^{\mu\nu\sigma} b_\mu \partial_\nu a_\sigma \\
&= \frac{k}{2\pi}\int d^3x\,[\varepsilon^{ij}\dot{a}_i b_j + a_0(\varepsilon^{ij}\partial_i b_j) + b_0(\varepsilon^{ij}\partial_i a_j)]\,.
\end{aligned} \tag{1.215}
$$

Note that although the Hamiltonian formally vanishes in the $a_0 = b_0 = 0$ gauge, these fields are Lagrange multiplier fields that impose the "Gauss law" constraints $\varepsilon^{ij}\partial_i a_j = \varepsilon^{ij}\partial_i b_j = 0$, which can be solved by,

$$
\begin{aligned}
a_i &= \partial_i \lambda_a(\mathbf{x},t) + \frac{2\pi}{L_i}\bar{a}_i(t) \\
b_i &= \partial_i \lambda_b(\mathbf{x},t) + \frac{2\pi}{L_i}\bar{b}_i(t)
\end{aligned} \tag{1.216}
$$

where $\bar{a}_i$ and $\bar{b}_i$ are spatially constant, and $\lambda_{a/b}$ are periodic functions on the torus. Inserting this into (1.215) gives the Lagrangian

$$
S_{BF} = k2\pi \int dt\,\varepsilon^{ij}\dot{\bar{a}}_i \bar{b}_j\,. \tag{1.217}
$$

From this we can read the canonical commutation relations[17].

$$
[\bar{a}_i, \bar{b}_j] = \frac{i}{2\pi k}\varepsilon^{ij}\,. \tag{1.218}
$$

The Wilson lines along the cycles of the torus given by,

$$
\mathscr{A}_i = e^{i\oint dx_i\, a_i} = e^{2\pi i \bar{a}} \qquad \mathscr{B}_i = e^{i\oint dx_i\, b_i} = e^{2\pi i \bar{b}} \tag{1.219}
$$

are invariant under the large gauge transformations $\bar{a}_i \to a_i + n_i^a$, and $\bar{b}_i \to b_i + n_i^b$, corresponding to threading unit fluxes though the holes in the torus.

To assume that the charges that couple to the gauge fields a and b are conserved is equivalent to taking the these fields to be compact, which means that field configurations differing by the large gauge transformations are identified. Put differently, the dynamical variables are not the constant $U(1)$ fields $\bar{a}_i$ and $\bar{b}_i$, but the Wilson loop operators $\mathscr{A}_i$ and $\mathscr{B}_i$, which satisfy the commutation relations,

$$
\mathscr{A}_l \mathscr{B}_m - e^{\frac{2\pi i}{k}} \mathscr{B}_m \mathscr{A}_l = 0 \quad l \neq m\,. \tag{1.220}
$$

For $k = 1$ all operators commute, and there is a unique ground-state, while for $k = 2$ we have the algebra,

[17] In case you do not know how to handle actions that are first order in time derivatives, you can learn in e.g. [65].

$$\mathscr{A}_x\mathscr{B}_y+\mathscr{B}_y\mathscr{A}_x=0 \quad \text{and} \quad \mathscr{A}_y\mathscr{B}_x+\mathscr{B}_x\mathscr{A}_y=0\,. \tag{1.221}$$

Each of these algebras have a two dimensional representation (think of the Pauli matrices!) and thus the ground-state is $2\times 2=4$ fold degenerate.

1.8.3 The relation between the Chern number and the Pontryagin index

We will use the notation, $\hat{d}(\mathbf{k})\cdot\sigma\,\xi_\pm(\mathbf{k})=\mp\xi_\pm(\mathbf{k})$ and $\xi\equiv\xi_+$ which means that the creation operator for the filled band takes the form

$$a^\dagger_{\mathbf{k}}=\sum_\alpha \xi^\alpha_-(\mathbf{k})c^\dagger_{\mathbf{k},\alpha} \tag{1.222}$$

and using the definition (1.55) the Berry connection is

$$\mathscr{A}_i=-i\{a_{\mathbf{k}},\partial_{k_i}a^\dagger_{\mathbf{k}}\}=-i\xi^\dagger(\mathbf{k})\partial_{k_i}\xi(\mathbf{k}) \tag{1.223}$$

Now suppressing the $\mathbf{k}$ dependence and using the short hand notation $\partial_{i/j}\equiv\partial_{k_{i/j}}$ we have,

$$\begin{aligned}(\partial_i\xi^\dagger)\partial_j\xi&=\partial_i(\xi^\dagger\hat{d}\cdot\sigma)\partial_j\xi=(\partial_i\xi^\dagger)\hat{d}\cdot\sigma\partial_j\xi+\xi^\dagger(\partial_i\hat{d}\cdot\sigma)\partial_j\xi\\&=(\partial_i\xi^\dagger)\hat{d}\cdot\sigma\partial_j\xi+\xi^\dagger(\partial_i\hat{d}\cdot\sigma)\partial_j(\hat{d}\cdot\sigma\xi)\\&=(\partial_i\xi^\dagger)\hat{d}\cdot\sigma\partial_j\xi-\xi^\dagger(\partial_i\hat{d}\cdot\sigma)\partial_j\xi+\xi^\dagger(\partial_i\hat{d}\cdot\sigma)(\partial_j\hat{d}\cdot\sigma)\xi\end{aligned} \tag{1.224}$$

where we in the last equality used $\sigma^a\sigma^b=-\sigma^b\sigma^a+2\delta^{ab}$ and $\hat{d}\cdot\partial_i\hat{d}=0$. Making similar manipulations we get,

$$(\partial_i\xi^\dagger)\partial_j\xi=(\partial_i\xi^\dagger)\hat{d}\cdot\sigma\partial_j\xi+\frac{1}{2}\xi^\dagger(\partial_i\hat{d}\cdot\sigma)(\partial_j\hat{d}\cdot\sigma)\xi \tag{1.225}$$

Next winvokee need the spectral decomposition of the Hamiltonian, $\hat{d}\cdot\sigma=\xi\xi^\dagger-\xi_-\xi_-^\dagger$ and the resolution of unity $1=\xi\xi^\dagger+\xi_-\xi_-^\dagger$ to get the two identities,

$$\begin{aligned}(\partial_i\xi^\dagger)\hat{d}\cdot\sigma\partial_j\xi&=(\partial_i\xi^\dagger)(\xi\xi^\dagger-\xi_-\xi_-^\dagger)\partial_j\xi=-\mathscr{A}_i\mathscr{A}_j+(\xi^\dagger\partial_i\xi_-)(\xi_-^\dagger\partial_j\xi)\\(\partial_i\xi^\dagger)\partial_j\xi&=(\partial_i\xi^\dagger)(\xi\xi^\dagger+\xi_-\xi_-^\dagger)\partial_j\xi=-\mathscr{A}_i\mathscr{A}_j-(\xi^\dagger\partial_i\xi_-)(\xi_-^\dagger\partial_j\xi)\end{aligned} \tag{1.226}$$

where we used $\xi^\dagger\partial_i\xi=-(\partial_i\xi^\dagger)\xi$ *etc.*. Combining (1.225) and (1.226) gives,

$$(\partial_i\xi^\dagger)\partial_j\xi=-\mathscr{A}_i\mathscr{A}_j+\frac{1}{4}\xi^\dagger(\partial_i\hat{d}\cdot\sigma)(\partial_j\hat{d}\cdot\sigma)\hat{d}\cdot\sigma\xi \tag{1.227}$$

Again using the σ matrix algebra, we finally get,

$$\mathscr{B}(\mathbf{k}) = i\varepsilon_{ij}\partial_i\xi^\dagger\partial_j\xi = -\frac{1}{2}(\partial_i\hat{d}\times\partial_j\hat{d})\cdot\hat{d} \tag{1.228}$$

1.8.4 The parity anomaly in 2+1 dimensions

The most direct way to extract the effective action (1.68) is to simply calculate the Feynman diagram in Fig.1.3 with a suitable regularization [36]. An alternative, and quite instructive, way is to pick a specific background field where the Dirac equation can be solved exactly and then invoke gauge and Lorentz invariance to find the general result.

We define the Dirac α-matrices by,

$$(\beta,\alpha^1,\alpha^2) = (\sigma^3,-\sigma^2,\sigma^1) \tag{1.229}$$

and will use complex coordinates

$$z = \sqrt{\frac{eB}{2}}(x+iy) \tag{1.230}$$

and the notation $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. In the symmetric gauge, $\mathbf{A} = \frac{B}{2}(-y,x)$, where B is a constant magnetic field. The Hamiltonian for the relativistic massless Landau problem becomes,

$$\begin{aligned} H &= \alpha\cdot(\mathbf{p}-e\mathbf{A}) \qquad (1.231)\\ &= \sqrt{eB}\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}(\partial-\bar{z}) \\ \frac{1}{\sqrt{2}}(\bar{\partial}+z) & 0 \end{pmatrix} = \sqrt{eB}\begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix} \end{aligned}$$

where $\mathbf{p} = -i\nabla$, and $[a,a^\dagger] = 1$. Introducing the corresponding number operator state $a^\dagger a\,|n\rangle = n\,|n\rangle$, we easily find the following solutions to the Schrodinger equation,

$$|\Psi_0\rangle = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \quad ; \quad E_0 = 0 \tag{1.232}$$

$$|\Psi_{n\pm}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} |n\rangle \\ \pm\,|n-1\rangle \end{pmatrix} \quad ; \quad E_\pm = \pm\sqrt{neB}\,. \tag{1.233}$$

Note that after adding a mass term, $H_m = \beta m$, $|\psi_0\rangle$ is still a solution but with the eigenevalue $E_0 = m$.

The energy levels are however massively degenerate, and in the radial gauge the relevant extra quantum number is the angular momentum. Defining,

$$b^\dagger = a + \sqrt{2}z \tag{1.234}$$

we have $[b,b^\dagger] = [a,a^\dagger] = 1$ with all other commutators vanishing, so the full Schrodinger spectrum is obtained from (1.232), by the replacement

$$|n\rangle \to \frac{(b^\dagger)^k}{\sqrt{k!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0,0\rangle \,. \tag{1.235}$$

Next we expand the Dirac field operator in the eigenfunctions $\psi_{n,k\pm} = \langle x\rangle\, \Psi_{k,n\pm}$

$$\hat{\psi}(x) = \sum_{n,k} \left[\psi_{n,k+} e^{-iE_n t} c_{n,k} + \psi_{n,k-} e^{iE_n t} d^\dagger_{n,k} \right] + \sum_k \psi_{0,k} e^{-imt} e \tag{1.236}$$

where we regularized the zero-mode by adding a small mass $m^2 \ll eB$, and introduced the fermionic operators $c_{n,k}$ and $d_{n,k}$ satisfying $\{c^\dagger_{\tilde{n},\tilde{k}}\} = \delta_{n,\tilde{n}}\delta_{k,\tilde{k}}$ *etc.*, and the Majorana operator e which satisfies $e^2 = 0$ and anti-commutes with all other Fermi operators.

For simplicity we considered the $m = 0$ case, but it is not too hard to find the full solution even for $m \neq 0$. The zero-modes we have already found, and the only thing we shall need for the following is that the rest of the spectrum is gapped and symmetric around $E = 0$ which follows from charge conjugation symmetry; fact the energies are $E_n = \sqrt{neB + m^2}$.

The next step is to calculate the vacuum expectation value of the current operator

$$j^\mu = \frac{e}{2}[\bar{\psi}, \gamma^\mu \psi] \tag{1.237}$$

and for simplicity we shall just consider the time component, i.e. the charge. Realling that $\gamma^0 = \beta$, we get

$$j^0 = \rho = \frac{e}{2}(\psi^\dagger\psi - \psi\psi^\dagger) \tag{1.238}$$

and because of the charge conjugation symmetry, only the zero-modes contribute to the expectaton value

$$\langle\rho\rangle = \pm\frac{e}{2}\sum_k |\psi_{0,k}|^2 \,. \tag{1.239}$$

where the positive sign corresponds to a negative m meaning that the (almost) zero-modes are all filled so the contribution comes from the first term in (1.238); the negative sign amounts to these modes all being empty. A direct calculation of the zero-mode wave functions gives,

$$\sum_k |\psi_{0k}|^2 = \sum_k |\sqrt{\frac{eB}{2\pi}}\sqrt{\frac{2^k}{k!}} z^k e^{-|z|^2}|^2 = \sum_k \frac{eB}{2\pi}(2|z|^2)^n e^{-|z|^2} = \frac{eB}{2\pi} \tag{1.240}$$

which just shows that the density of state in the lowest Landau level is the same as in the non-relativistic case. Inserting this in (1.238) gives

$$\langle \rho \rangle = \frac{m}{|m|} \frac{e^2 B}{4\pi} \tag{1.241}$$

and by Lorentz and gauge invariance we get

$$\langle j^\mu \rangle = \frac{m}{|m|} \frac{e^2}{4\pi} \varepsilon^{\mu\nu\sigma} \partial_\nu A_\sigma = -\frac{\delta \Gamma_D[A]}{\delta A_\mu}, \tag{1.242}$$

for any constant electric, or magnetic, field. Integrating this expression to get the effective response action, we get the result (1.68) quoted in the main text. Note that although the Chern-Simons term is not gauge invariant, its variation, given by (1.242), is.

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