UNIVERSAL SCALING IN CIRCLE MAPS

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We discuss scaling, renormalization, and universality for critical and subcritical circle maps. Using the structure of the rationals provided by the Farey tree, we construct a pair of renormalization group operators for circle maps with all values of the winding number. In this general formulation, the renormalization equations define a universal invariant set which determines all universal quantities for all cubic maps with all irrational winding numbers. As expected, the manifold of pure rotations defines the subcritical attracting set. We show that the invariant set governing the critical circle maps cannot be embedded in less than 3 dimensions. However, we are able to obtain a differentiable parametrization of the unstable manifold of the critical invariant set, which we believe is universal up to Lipschitz homeomorphisms; the derivative of this function describes the scaling structure of all small gaps in the devil's staircase for all critical maps. Our work can be viewed as providing numerical evidence for Lanford's conjectures on the strange set underlying renormalizations of critical circle maps.

1. Introduction

1.1. History of the problem

A large effort has been made in recent years to understand the transition to chaos in quasi-periodic dissipative systems. One of the most successful approaches has been the use of the renormalization group to study high iterates of maps of the circle onto itself [1, 2]. From this has emerged a fairly complete picture of scaling and universality of maps with special winding numbers, such as the golden mean. However, an intriguing question has remained about universal properties for more general irrationals. The present paper reports on our efforts to understand this general problem.

A strength of the renormalization group (RG) is that fixed points of a relatively simple RG recursion formula determine various universal properties. A fixed point is a simple example of an invariant set; when no RG fixed point exists and the renormalization trajectory ends up on a more complicated invariant set, it will be this invariant set which will determine universal properties. Previous RG work has shown that under renormalization maps with quadratic irrationals iterate to an RG fixed point [1, 2].

The RG invariant set for subcritical circle maps (C^\infty diffeomorphisms) is a collection of maps which can be thought of as a one-dimensional manifold. In the present paper, we focus on the properties of an invariant set under RG transformations for critical circle maps, which we call the critical set [3]. Our major result is to show that the critical set leads to the existence of a universal function whose derivative determines universal scaling. Higher derivatives of this function are singular and non-universal. We also show to what extent this invariant set can be parametrized by smooth coordinates.

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Two crucial facts underlie the present work. The first is the realization that two different circle maps with the same but arbitrary irrational winding number have trajectories which converge during renormalization. This was first pointed out by Ostlund et al. [2]. The second is that at any stage in the renormalization process a pair of strings can be used to label the location along a RG trajectory. A tail string labels the future trajectory and a head string labels the history. We attribute this idea to MacKay [4], who used it to analyze the boundary of Siegel disks [5]. The present paper describes our attempts to implement and quantify these ideas.

The same ideas have been used by Lanford to study the critical set. He has proved that under suitable hypotheses a continuously differentiable parametrization of the unstable manifold of the critical set exists such that the induced action is affine. Although we originally obtained our results independently, the present work can now be viewed as providing numerical confirmation of the picture of the critical set that was more rigorously described by Lanford [6]. In our present work we focus on the pictorial representation of the critical set and a description of its topological and metric properties.

Several authors have explored similar questions. Pictorial cross-sections of the invariant set and a computation of average scaling exponents that are independent of winding number have been considered by Farmer, Satija, and Umberger [7, 8]. These authors also provided evidence that the critical set could be "strange." David Rand recently showed how to obtain a universal description of such RG invariant sets [9]. Our picture of the critical set is consistent with these pictures, and we hope to provide a more detailed, primarily numerical, exploration of the critical set than has been provided before.

1.2. Fundamental concepts

Although the intent of the present work is to study dissipative dynamical systems, we shall in fact work with maps of the circle. It is widely believed that these functions display the same universal behavior as the continuous dynamical system. The circle map is represented by a real function \( f(x) \) obeying \( f(x + p) = p + f(x) \) when \( p \) is integer. We shall restrict ourselves to the case when \( f(x) \) is monotonic and all derivatives of \( f \) exist. Successive applications of \( f \) generate a sequence of \( x_n \) defined by

\[
x_{n+1} = f(x_n).
\]

Given our constraints [10,11] on \( f \), the winding number \( \rho(f) \) can be uniquely defined by

\[
\rho(f) = \lim_{n \to \infty} \frac{x_n - x_0}{n}.
\]

The standard sine map is a 2-parameter family of functions defined by

\[
f_{a,\Omega}(x) = x + \Omega - \frac{a}{2\pi} \sin(2\pi x).
\]

We usually focus attention on a particular value of \( a \), in which case we write \( f_{\Omega} = f_{a,\Omega} \).

When the inverse function \((f_{\Omega})^{-1}\) is \(C^\infty\) \((f_{\Omega} \) is a diffeomorphism) the generic behavior of high iterates of \( f \) can be completely understood by the fact that for most [12] irrational winding numbers, \( \rho(f_{\Omega}) \), \( f_{\Omega} \) can be obtained from the pure rotation \( R(x) = x + \omega \) where \( \omega = \rho(f_{\Omega}) \) via an infinitely differentiable coordinate transformation [10,11]. For the standard sine map this includes the entire region \( a < 1 \).

When \( a > 1 \) the standard sine map ceases to be invertible and dynamics are much more complicated [13]. If we fix \( \Omega \) to maintain a fixed irrational winding number \( \rho(f_{\Omega}) \), a chaos transition occurs as \( a \) passes through 1. We refer to \( a = 1 \) as the critical value of the nonlinearity and \( a < 1 \) as the subcritical regime. Previous renormalization group (RG) work [1, 2] has been able to describe
the transition at $a = 1$ in some detail for special values of $\rho$.

Let us consider the standard sine map $f_\Omega$. A plot of $\rho(f_\Omega)$ as a function of $\Omega$ (fig. 1) reveals an infinite series of steps corresponding to intervals in $\Omega$ where $\rho(f_\Omega)$ locks to a rational number [14]. Because of the existence of infinite steps of ever finer scales this function is called the devil's staircase.

A schematic phase diagram is shown in fig. 2. Regions which are shaded correspond to parameter values of $a$ and $\Omega$ where $\rho(f_{a,\Omega}) = p/q$ is locked to a few low order rational numbers. These regions are referred to as phase locked tongues. The entire white region contains an infinity of tongues which are not shown. As the denominator $q$ grows, the tongues shrink to zero width and the demand that $\rho(f_{a,\Omega})$ be irrational yields a line such as one shown in the figure.

For the circle map, it is known that for fixed $a < 1$ the phase locked regions occupy less than full measure, i.e.

$$\lim_{q_{\text{max}} \to \infty} \sum_{q = 1}^{q_{\text{max}}} \sum_{p(q)} \Delta(p/q) < 1. \quad (1.4)$$

where $\Delta(p)$ is the width of the step in $\Omega$ at the particular value of $\rho = \rho(f_\Omega)$ and the summation $p(q)$ denotes a sum over the integers $p$ in $[0, q - 1]$ which are relatively prime to $q$. The values of $\Omega$ corresponding to irrational winding numbers at fixed $a$ occupy a set of finite measure. At $a = 1$, it has been shown, first numerically [14] and later rigorously [15], that eq. (1.4) becomes an equality, so that the values of $\Omega$ corresponding to irrational values of $\rho$ occupy zero measure.

Both numerical and semi-rigorous RG work indicate that certain properties of the fine scale structure in the devil's staircase are universal. For example, the fractal dimension of the devil's staircase has been numerically measured and found to be independent of a particular choice of 1-parameter family of maps [14]. Certain ratios of gap sizes near winding numbers corresponding to irrational winding numbers which are roots of quadratic equations with integer coefficients can almost certainly be proved to be universal using rigorous renormalization group arguments. Near values of $\rho$ corresponding to $(n\alpha_3 + m)/(p\alpha_3 + q)$ where $n$, $m$, $p$ and $q$ are integers satisfying
In the next section, we review known results on the scaling and renormalization of the circle map using the continued fraction representation to organize a RG structure. Although the present work relies heavily on this picture, we discuss why the continued fraction structure is not completely satisfactory. In section 3, we introduce a Farey tree organization of rationals [17–19], and discuss the binary labeling schemes of the base 2 decimals as well as Farey rationals. The relation between the binary Farey tree and continued fractions is illustrated. In section 4, we redefine the renormalization group using the Farey tree to organize its structure. In section 5, we discuss the parametrization of the critical set, the set of functions which are stable and invariant under this renormalization. In section 6, we discuss some conjectures regarding the analytic structure of the Devil's staircase. In section 7, we explore the relation between the scaling function and the scaling structure of small gaps in the devil's staircase. We make a numerical approximation to the continuous and differentiable scaling function, generating a mock devil's staircase which mimics the devil's staircase of the critical problem accurately.

2. Review of renormalization group concepts

2.1. Scaling functions and exponents

We begin by reviewing well-known concepts of number theory and renormalizations for circle maps. The continued fraction representation of an irrational number \( \rho \) is denoted by

\[
\rho = [n_0, n_1, n_2, \ldots]_{\text{cf}}
\]

The rational number \( \rho_k = [n_0, n_1, \ldots, n_k]_{\text{cf}} \) is called the \( k \)th rational approximant to \( \rho \).

An important special subclass of irrational numbers are called \textit{quadratic}, or \textit{periodic}. These irrational numbers have tails in their continued fraction representation which eventually become periodic. That is, for some \( N \) and for every \( k > N \), there is a period \( L \) and integer \( N \) such that \( n_{k+L} = n_k \) for every \( k > N \). This is equivalent to \( \rho \) satisfying a quadratic equation with integer coefficients [20].

For quadratic irrational winding numbers, a set of universal quantities exists for both critical and subcritical maps. Assume that \( \rho(f_{\Omega_*}) \) is a quadratic irrational whose tail in the continued fraction representation is of period \( L \). Then the functions \( \xi_j^*(x) \) for \( 0 \leq j < L \) defined by [2]

\[
\xi_j^*(x) = \lim_{n \to \infty} a^{(nL+j)} (f_{\Omega_*}^n (x/a^{(nL+j)}) - p_{nL+j})
\]

exist and are universal, i.e., independent of the choice of one parameter family of maps \( f_{\Omega_*} \). The quantity \( a^{(nL+j)} \) is given by

\[
a^{(k)}(f_{\Omega_*}^n(0) - p_k - f_{\Omega_*}^{q_k+q}(0) + p_{k+1})^{-1},
\]

where

\[
p_k/q_k = \rho_k,
\]

and \( p_k \) and \( q_k \) are relatively prime integers and \( \Omega_* = \lim_{k \to \infty} \Omega(\rho_k) \), where \( \Omega(\rho_k) \) is the parameter value corresponding to the winding number \( \rho \).
The notation \( f_{q}^{*} \) means that \( f_{q} \) composed with itself \( q \) times. The \( \alpha^{(k)} \) defines a number \( \alpha \), called the domain scaling exponent, by

\[
\lim_{k \to \infty} \frac{\alpha^{(k)}}{\alpha^{(k-L)}} = (-1)^{L} \alpha, \tag{2.5}
\]

where \( |\alpha| > 1 \). The number \( \alpha \) depends only on the finite repeating pattern for the tail of the entries in the continued fraction representation and is universal in that sense. We define a continued fraction parameter scaling exponent \( \delta_{cf}(\rho) \) by

\[
\delta_{cf}(\rho) = \lim_{k \to \infty} \frac{\Omega_{*} - \Omega(\rho_{kL+j})}{\Omega_{*} - \Omega(\rho_{(k+1)L+j})}, \tag{2.6}
\]

where \( \Omega(\rho) \) is the parameter value corresponding to the winding number \( \rho \). This number is independent of \( j \) and numerically equivalent to

\[
\delta_{cf}(\rho) = (-1)^{L} \lim_{k \to \infty} \frac{\Delta(\rho_{kL+j})}{\Delta(\rho_{(k+1)L+j})}, \tag{2.7}
\]

where \( \Delta(\rho) \) is the width of a phase locked interval in \( \Omega \) for the winding number \( \rho \). The numerical equality of the two definitions can be understood by RG arguments.

For a given periodic tail in the continued fraction, there are distinct universal values of \( \delta_{cf} \) depending only on whether the map is critical or subcritical. From eq. (2.6) the value of \( \delta_{cf} \) for the subcritical maps can be obtained by number theory and standard functional methods using the one-parameter family of maps generated by pure rotations. The value of \( \delta_{cf} \) for critical maps must be numerically computed.

2.2. Construction of the reduced map

The reduced map corresponding to the function \( f \) maps a unit interval to itself and is equivalent to \( f \) modulo integers. We define a general reduced map \([\xi, \eta]\) via a pair of functions \( \xi \) and \( \eta \) which obey [21]

\[
\xi(0) > 0, \eta(0) < 0, \xi'(x) \geq 0, \eta'(x) \geq 0, \xi \eta(0) = \eta \xi(0), \xi(0) - \eta(0) = 1. \tag{2.8}
\]

The reduced map \([\xi, \eta]\) acts on the interval \( \eta(0) < x < \xi(0) \) by

\[
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
= \begin{bmatrix}
\xi(x) & \text{if } \eta(0) < x < 0, \\
\eta(x) & \text{if } 0 < x \leq \xi(0)
\end{bmatrix}. \tag{2.9}
\]

The reduced map equivalent to the circle map \( f \) is

\[
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
= \begin{bmatrix}
f \\
f - 1
\end{bmatrix}. \tag{2.10}
\]

Critical maps obey \( \xi'(0) = \xi''(0) = \eta'(0) = \eta''(0) = 0 \). A typical critical map is shown in fig. 3.

A proper or commuting reduced map is a reduced map for which the functions \( \xi \) and \( \eta \), when they are analytically continued in a small region outside their natural domains, commute under composition; there exists an \( \varepsilon \) that for \( |x| < \varepsilon \)

\[
\xi \eta(x) = \eta \xi(x). \tag{2.11}
\]

It is easy to check that the reduced map in eq. (2.10) is proper.

Let \( D(\xi), R(\xi), D(\eta) \) and \( R(\eta) \) be the domains and ranges of \( \xi \) and \( \eta \), respectively. By...
inspecting fig. 3, (if necessary turn the page upside down and relabel the functions and axes) it can be seen that unless $\xi(\eta(0)) = 0$ one of the following conditions labeled 0 or 1 must hold:

condition (0): $R(\eta) \subseteq D(\xi)$ iff $\xi(0) < 0$,

\[ (2.12a) \]

condition (1): $R(\xi) \subseteq D(\eta)$ iff $\xi(0) > 0$.

\[ (2.12b) \]

If $\xi(0) = 0$, it implies that the origin is a 2-cycle. In this case, the renormalization operator is left undefined.

2.3. Continued fraction renormalization group

Let us assume that $\xi(0) < 0$ as in fig. 3 and that $[\xi, \eta]$ has a superstable [22] $p/q$ cycle. Label the elements of the cycle from left to right $x_j$, where the left hand side $x$ is $x_{-q+p}$, the origin $x_0$, and the right hand side $x_p$. Since the left and right endpoints are identified, $x_{-q+p}$ and $x_0$ correspond to the same cycle element. It can be verified that $[\xi, \eta](x_j) = x_{(j+p)_q}$ where $(j+p)_q$ means $j + p$ modulo the integer $q$. Since condition (0) in eq. (2.12) holds, we can define an integer $n > 0$, for which

\[ \xi^{n-1}\eta(0) < 0, \quad \xi^{n-1}\eta\xi(0) > 0. \]  \[ (2.13) \]

If $\xi(0) > 0$, that is, condition (1) holds then take $n = 1$. Now we define the number $\alpha_n$ by

\[ \alpha_n = (\xi^{n-1}\eta(0) - \xi^{n-1}\eta\xi(0))^{-1} < -1. \]  \[ (2.14) \]

The renormalization group operator $T_n$ is defined by

\[ T_n \begin{bmatrix} \xi \\ \eta \end{bmatrix}(x) = \alpha_n \begin{bmatrix} \xi^{n-1}\eta \\ \xi^{n-1}\eta\xi \end{bmatrix}(x/\alpha_n) = \begin{bmatrix} \xi_R \\ \eta_R \end{bmatrix}(x). \]  \[ (2.15) \]

The following statements are then known to be true [22]:

1) If $[\xi, \eta]$ is proper, then $[\xi_R, \eta_R]$ is proper.

\[ \rho(T_n([\xi, \eta])) = \frac{1}{\rho([\xi, \eta])} - n. \]  \[ (2.16) \]

Property (2) shows that $T_n$ is a renormalization group operator which removes the first element in the continued fraction representation of the winding number. Property (1) shows that the proper reduced maps are closed under $T_n$.

2.4. Results of the continued fraction renormalization group

The action of $T$ in the space of proper reduced maps has been studied in detail when $\rho([\xi, \eta])$ is a quadratic irrational, especially when $\rho = \sigma_c = \frac{1}{2}(\sqrt{5} - 1) = [1,1,1,\ldots]_c$. Assume that $\rho = [n_0, n_1, n_2, \ldots]_c$, where $n_{k+1} = n_k$ for $k > N$, where $N$ is some large integer. Define the operator $T_{\text{head}}$ and $T_{\text{period}}$ by

\[ T_{\text{head}} = T_{n_0}T_{n_{N-1}} \cdots T_{n_2}, \]

\[ T_{\text{period}} = T_{n_{N+1}} \cdots T_{n_{N-1}}. \]  \[ (2.17) \]

We then know the following [2]:

1) The limit

\[ \lim_{m \to \infty} (T_{\text{period}})^m(T_{\text{head}}) \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi^* \\ \eta \end{bmatrix} \]  \[ (2.18) \]

exists and $\xi^*$ is in fact identical to one of the $\xi_j^*$ in eq. (2.2).

2) There is a solution $[\xi, \eta]_j^*$ to each of $L$ fixed point equations for

\[ \text{Perm}_j(T_{\text{period}}) \begin{bmatrix} \xi^* \\ \eta \end{bmatrix} = \begin{bmatrix} \xi^* \\ \eta \end{bmatrix}, \quad j = 1,\ldots, L, \]  \[ (2.19) \]

where $\text{Perm}_j(T_{\text{period}})$ is defined to be $j$ cyclic permutations of component operators of $T_{\text{period}}$ in eq. (2.17). It can be shown that the two
Table I
A summary of analyzed periodic continued fractions. Universality with respect to all circle maps has been derived by an explicit numerical analysis of the fixed point equations for these values of $p$.

<table>
<thead>
<tr>
<th>Address</th>
<th>$\rho$</th>
<th>$\alpha$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,1,1,...</td>
<td>$(\sqrt{2} - 1)/2$</td>
<td>-1.28862</td>
<td>-2.83361</td>
</tr>
<tr>
<td>2,2,2,...</td>
<td>$\sqrt{2} - 1$</td>
<td>-1.5868</td>
<td>-6.79925</td>
</tr>
<tr>
<td>1,2,1,...</td>
<td>$\sqrt{3}$</td>
<td>1.140832</td>
<td>17.66906</td>
</tr>
</tbody>
</table>

definitions of $\xi_f^*$ in eqs. (2.2) and (2.19) are equivalent.

3) The exponent $\delta_c(\rho)$ in eq. (2.6) is identical to the leading eigenvalue of the Jacobian of $T_{period}$ at the fixed point.

This picture has been analyzed in detail for a particular set of periodic irrationals shown in table I [2]. Universality of scaling shown in table I has been derived by the explicit numerical analysis of the fixed point equations for these values of $p$. The computer-assisted proof of the existence of the critical golden mean fixed point has been provided by Mestel [16].

This completes our overview of relevant facts which are known about the scaling structure of the circle map. Many technical details have been glossed over regarding the existence of relevant and marginal but physically inaccessible eigenvalues at the renormalization group fixed point. Universality with respect to embedding the mapping in higher dimension has been discussed in ref. [2], and more rigorously by Rand [23].

2.5. Why this picture is not complete

The renormalization group based on the continued fraction suffers from a few undesirable restrictions. Only when $\rho$ is a quadratic irrational can we analyze a fixed point. In spite of this restriction, there is evidence indicating the existence of a more global structure with respect to an arbitrary winding number. We cite this evidence.

1) Let $\rho(f) = \rho(g) = [n_0, n_1, \ldots]_c$, where $f$ and $g$ are two critical maps. Then (conjecture "D" in ref. [2])

$$\lim_{k \to \infty} \left\| T_{n_k} T_{n_{k-1}} \cdots T_{n_0} \begin{bmatrix} f \\ \rho(f - 1) \end{bmatrix} \right\| = 0$$

even for a random sequence of $n_k$, where $\| \|$ is a metric which measures the distance between two points in the function space. (All reasonable metrics work.)

2) For the subcritical maps, successive renormalization always iterate to pure rotations. Pure rotations can thus be considered an invariant manifold which is stable under $T_n$ [2, 10, 11].

3) Various fractal exponents which probe global (winding number independent) properties of the devil's staircase have been numerically measured and found to be universal [14, 7, 8].

Since the $n$ in $T_n$ (eqs. (2.14), (2.15)) is a discontinuous functional of the argument $[x, y]$, the form of the RG operator varies discontinuously over the space of proper reduced maps. Furthermore, since the range of each operator intersects the domain of every operator $T_n$ in the space of proper reduced maps, an infinite number of coupled equations must be simultaneously solved to obtain the invariant set under renormalizations directly from the fixed point equations. This approach seems very difficult to carry through. A further, perhaps aesthetic objection to the continued fraction structure is that the symmetry $\rho \rightarrow 1 - \rho$ which is inherent in the problem is violated by the continued fraction renormalization operations.

The Farey organization of rationals, called the Farey tree, overcomes these problems. Every result obtained from the continued fractions can be recovered by the Farey algorithm. However, the Farey tree enables us to explore more detailed scaling structure such as scaling associated with the sequence of winding numbers given by $1/n$. $n$
integer. For this reason, the Farey algorithm has been frequently used in analyzing the scaling observed in many mode-locking experiments, and it is the approach we will use in this paper also. We develop the number theoretic machinery in the next section before explaining consequences for renormalization in section 4.

3. Number theory of the Farey tree

3.1. Definition of the Farey tree

The Farey tree [17–19] represents an organization of rationals, \( p/q \), where \( p \) and \( q \) are relatively prime integers. The structure of the tree is imposed by the Farey sum [24], or mediant operations, \( \circ \), which interpolates between two rationals, \( p/q \) and \( p'/q' \), such that 

\[
\frac{p}{q} \circ \frac{p'}{q'} = \frac{p + p'}{q + q'}.
\]

The resulting rational is called a mediant and falls between two parent rationals. The entire tree can be constructed from two rationals, \( 0/1 \) and \( 1/1 \), by the recursive applications of the mediant operation on the nearest neighbor pairs of rationals.

Define \( 0/1 \) and \( 1/1 \) to be two level 1 elements of the tree. Then the mediant operation on these two elements generates the level 0 Farey element, \( 1/2 = 0/1 \circ 1/1 \). Now we apply the mediant operation on two numerically close pairs of rationals, so that \( 0/1 \circ 1/2 \) and \( 1/2 \circ 1/1 \) generate two level 1 elements, \( 1/3 \) and \( 2/3 \), respectively. The next application of the mediant operation will generate four level 2 Farey elements. This process can be repeated indefinitely to generate the entire Farey tree. Only the left hand side of the tree up to level 5 is shown in fig. 4.

The following facts are well established for the Farey tree.

i) All rationals \( p/q \) in \([0,1]\) (\( p, q \) relatively prime) appear exactly once in the tree.

ii) A number is in the level \( n \) if and only if the sum of entries in the continued fraction is \( n + 2 \).

3.2. Binary labeling schemes

It is important to note that Farey elements form a tree with a natural binary labeling, since every element in level \( n \) can be associated with two daughter elements. Bonds are drawn from the mother to the two daughters; the bond is labeled 0 if it is to the left and 1 to the right. (See fig. 4.) We define the binary address of \( 1/2 \) to be the empty string. The binary label of any element of the tree is read by the string of bond labels in the path from \( 1/2 \) to the given element [25]. Thus \( B_F(3/8) = [0, 1, 0] \), where \( B_F(x) \) is defined to be the binary Farey address (or simply the Farey address) of \( x \). It should be noted that the length of the string is crucial; trailing zeros cannot be dropped. Also note that there is a symmetry around \( 1/2 \) which can be expressed in terms of the Farey address by

\[
B_F(x) = [I_0, I_1, \ldots, I_n],
\]

\[
B_F(1-x) = [1 - I_0, 1 - I_1, \ldots, 1 - I_n].
\]  

The inverse operator \( b_F = B_F^{-1} \) which associates a number with the Farey address is defined by 

\[
x = b_F[I_1, I_2, \ldots, I_N].
\]

Thus 

\[
b_F[1/2] = 2/3, \quad B_F[0, 0, 1] = 2/7,
\]  

etc. The operator \( b_F \) is convergent in the sense...
that

\[ \limsup_{k \to \infty} N, b_F[I_0, \ldots, I_k, i_1 \ldots i_N] \]

\[ - b_F[I_0, \ldots, I_k, i_1 \ldots i_N] \leq 0. \]  (3.3)

The binary scheme of the Farey tree is natural. If we have replaced \( \oplus \) by the ordinary average \( (p/q + p'/q')/2 \), this scheme would generate the base 2 decimal binary tree, except that the last digit in the ordinary base 2 representation (which is always 1) is dropped. We will discuss the base 2 decimal binary tree before going into the binary Farey tree since many important facts about the tree can be learned from the simple base 2 decimal binary tree.

**3.2.1. Base 2 arithmetic**

In general, any real number \( x \) in the interval \([0, 1]\) can be written in terms of a base two representation.

\[ x = \frac{1}{2} I_0 + \left( \frac{1}{2} \right)^2 I_1 + \cdots + \left( \frac{1}{2} \right)^{n-1} I_n + \cdots \]  (3.4)

Assume that \( N \) (which could be infinite) is the smallest integer such that \( I_n = 0 \) for all \( n > N \). Then we can write

\[ x = (I_0 I_1 \cdots I_N)_{2}, \]  (3.5)

where subscript 2 denotes the base two representation of \( x \). The trailing zeros have been dropped. Then the binary address of \( x \) is defined to be

\[ B_2(x) = [I_0, I_1, \ldots, I_{N-1}]. \]  (3.6)

The last digit \( I_N \), which by our definition is always 1, is not included in the binary address.

Given a binary address, we can visualize the address as a location on a tree similar to the Farey tree in fig. 4. We can arrange numbers in such a way that the address can be obtained by a sequence of left and right bond labels traversed to reach the given number from the top of the tree. We define the parent of a node \( x \) to be the two nearest elements in the entire tree above \( x \). It can be shown that a daughter \( x \) is indeed the ordinary average of two parents, \( y \) and \( z \), i.e., \( x = (y+z)/2 \).

Given a number \( x \), there is an algebraic way of generating the binary address by simple recursive binary operation. For an ordinary base two binary, the shift operation \( t(x) \) which shifts the binary address to the left by one is just multiplication by two modulo 1. Thus \( t^{B_2}(x) \) is given by

\[ t^{B_2}(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2, \\ 2x - 1 & \text{if } 1/2 < x < 1. \end{cases} \]  (3.7)

The binary address of \( x \) relative to ordinary addition \( + \) is generated by the following recursion rules.

**Algorithm 1: Compute the binary address of \( x \)**

**Input:** \( x \)

**Output:** \([I_0, \ldots, I_N] = B_2(x) \)

**Initialize:** \( x_0 = x; \ k = 0 \)

\[ \text{do while } (x_k \neq 1/2) \{
\]

\[ \text{if } (x_k < 1/2) \ I_k = 0; \\
\text{if } (x_k > 1/2) \ I_k = 1; \\
x_{k+1} = t^{B_2}(x_k); \\
k \leftarrow k + 1; \\
\}

If \( x \) does not have a denominator which is a power of 2, \( k \) will be infinite since \( x_k \) never reaches \( 1/2 \).

**3.2.2. Binary Farey tree**

The preceding discussions are easily carried over to the Farey tree. Let us assume that \( p \) and \( q \) are positive integers with \( p > q \). We define the notation \( \{ \cdot \} \) by \( \{ p, q \} = p/(p+q) \). For \( p > 1/2 \) we see by inspecting fig. 4 that if

\[ \rho = \{ p, q \} = b_F[1, I_1, I_2, \ldots, I_k], \]  (3.8)
then
\[ b_F[I_1, \ldots, I_k] = \{ p - q, q \} = 2 - \frac{1}{\rho}. \tag{3.9} \]

For \( \rho < 1/2 \), if
\[ \rho = \{ p, q \} = b_F[0, I_1, I_2, \ldots, I_k], \tag{3.10} \]
then
\[ b_F[I_1, \ldots, I_k] = \{ p, q - p \} = \frac{p}{1 - p}. \tag{3.11} \]

We incorporate this into a definition. Define
\[ t_F(\rho) = \begin{cases} \frac{p}{1 - p} & \text{if } \rho < 1/2, \\ 2 - \frac{1}{\rho} & \text{if } \rho > 1/2. \end{cases} \tag{3.12} \]

Then
\[ t_F(b_F[I_0, I_1, \ldots]) = b_F[I_1, I_2, \ldots]. \tag{3.13} \]

The function \( t_F(\rho) \) removes the first element in the binary Farey address of \( \rho \). It is a function which commutes with reflection about 1/2 and is composed of two analytic pieces, rather than an infinity of pieces necessary for the analogous operator for the continued fraction representation. In that case \( t_F \) is replaced by the function \( 1/\rho - n \), where \( n \) is the integer part of \( 1/x \).

The operation \( B_F \) can be done by the following recursive algorithm.

Algorithm 2: Compute the Farey address of \( \rho \\
Input: \( \rho \) \\
Output: \( [I_0, \ldots, I_N] = B_F(\rho) \) \\
Initialize: \( \rho_0 = \rho; \ k = 0 \) \\
do while \( (\rho_k \neq 1/2) \) \\
\quad if \( (\rho_k < 1/2) \) \( I_k = 0; \) \\
\quad if \( (\rho_k > 1/2) \) \( I_k = 1; \) \\
\quad \rho_{k+1} = t_F^k(\rho_k); \ \\
\quad k \leftarrow k + 1; \)

For completeness, we write the inverse algorithm.

Algorithm 3: Compute \( \rho \) given the Farey address \\
Input: \( [I_0, \ldots, I_N] \) \\
Output: \( \rho = b_F[I_0, \ldots, I_N] \) \\
Initialize: \( \rho = 1/2; \ k = 0 \) \\
do while \( (k \leq N) \) \\
\quad if \( (I_k = 0) \) \( \rho = \rho/(1 + \rho); \) \\
\quad if \( (I_k = 1) \) \( \rho = 1/(2 - \rho); \) \\
\quad k \leftarrow k + 1; \)

If we define the symmetry operator \( r \) by \( r(x) = 1 - x \), then we get
\[ rr(x) = x, \quad rt_0^r = t_1^F, \quad rt_1^F r = t_0^F. \tag{3.14} \]

3.3. Conversion between continued fractions and the binary Farey address

The transformation between the binary Farey address and the continued fraction representation is not difficult to discover. Define the continued fraction shift operator \( t_n^c \) by \( t_n^c(\rho) = 1/\rho - n \). Then we get
\[ rt_1(t_0)^{n-1} = t_n^c. \tag{3.15} \]

Suppose that \( x = [n_0, n_1, \ldots, n_N] \), or equivalently,
\[ 0 = t_{n_N} \cdots t_{n_1} t_{n_0}(x), \tag{3.16} \]
then from eqs. (3.14) and (3.15), we get
\[ 0 = (t_1) \cdots \left[(t_0)^{n_1}(t_1)^{n_2}\right] \left[(t_0)^{n_3}(t_1)^{n_4}\right] \cdots \left[(t_0)^{n_{N-1}}(t_1)^{n_N}\right]. \tag{3.17} \]

Since we can always make the length of the continued fraction address even with the relation \( [n_0, \ldots, n_k] = [n_0, \ldots, n_k - 1, 1] \), we get the following transcription rule between the continued fraction and the Farey addresses:
\[ [n_0, \ldots, n_N] = [0^{n_0-1}(1^{n_0-1}0^{n_1})(1^{n_1-1}0^{n_2}) \cdots 1^{n_{N-1}}], \tag{3.18} \]
where \( N \) is even, and superscripts denote the number of repetitions of the enclosed binary address. Roughly speaking, if a continued fraction entry of, say, a sequence of 4 and 3 occurs in the
continued fraction representation of \( \rho \), the binary address will contain a sequence of four 1's followed by three 0's in the interior of the address. (The order of 0 and 1 may be reversed.) At either end of the string, this rule breaks down; we have to go back to eq. (3.18) for an exact correspondence.

4. Farey renormalization group

4.1. Goals of the renormalization group formulation

The previous discussion focused on the number theory associated with the rational approximants. The goal of our work is to develop a RG structure based on this number theory which can be used to study highly iterated maps of the circle. We hope to extend the ideas in ref. [2] to a complete description of all irrational winding numbers. We define four objectives which we consider desirable for a complete renormalization group formulation.

1) A structure must be imposed on real numbers which leads to a recursive formula for renormalizing the winding number.

2) A closed renormalization group operator must be constructed which is closed in some function space relevant to the problem and induces the transformation 1) on the winding number.

3) A set of closed functional fixed point equations must exist for the renormalization group which describes the behavior of high iterates of the original map.

4) We should be able to solve the fixed point equations to arbitrary accuracy, and relate eigenvalues of the linearized operator at the fixed point to the scaling structure and universality of the original problem.

Once objects 1)–4) have been achieved, it must be understood how the fixed point is embedded in higher dimensional spaces to understand the relevance of the fixed point to real physical systems.

For circle maps whose winding number is \( \sigma_G = (\sqrt{5} - 1)/2 \), these objectives have been achieved using the continued fraction renormalization group. For critical maps with non-quadratic winding numbers only objectives 1) and 2) have previously been accomplished, while 1)–3) and part of 4) (ref. [2] did not compute the contracting eigenvalues) have essentially been done for the subcritical problem.

Our goal has been to carry out the objectives 1)–4) for critical and subcritical circle maps. For each branch of the shift operator \( T_n^\sigma \), there is a different analytic form for the induced RG operation. The difficulty of dealing with an infinity of branches of the continued fraction renormalization operator necessary to achieve objective 3) for all irrational winding numbers caused us to seek a simpler renormalization group transformation which would involve as few branches of the RG operator as possible. We will reconstruct the renormalization operator using a transformation with only two branches. This simplification will enable us to carry out objectives 1)–3), and although we failed to carry out objective 4), we will discuss in section 5 why a direct solution of the fixed point equations for general winding numbers is unattainable.

4.2. Farey renormalization group transformation

We now discuss the functional generalization of eq. (2.15). Consider the proper reduced map \([\xi, \eta]\). Define the operator \( T[\xi, \eta] \) by

\[
T \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{cases} T_0 \begin{bmatrix} \xi \\ \eta \end{bmatrix} & \text{if } \xi \eta(0) > 0, \\ T_1 \begin{bmatrix} \xi \\ \eta \end{bmatrix} & \text{if } \xi \eta(0) < 0, \end{cases}
\]

(4.1)

where

\[
T_0 \begin{bmatrix} \xi \\ \eta \end{bmatrix} (x) = \alpha \begin{bmatrix} \xi \\ \xi \eta \end{bmatrix} (x/\alpha). \tag{4.2a}
\]

\[
\alpha = (\xi(0) - \xi \eta(0))^{-1}, \tag{4.2b}
\]
Note that $\alpha > 1$. (If $\xi(0) = 0$, we must stop the renormalization since we have renormalized to a function with a 2-cycle.) The reflection operator $R$ which connects $T_1$ and $T_0$ is defined by

$$ R \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] (x) = - \left[ \begin{array}{c} \eta \\ \xi \end{array} \right] (-x). \quad (4.4) $$

It is easy to see that $T_1 R T_0$ and $R R = \text{Identity}$. In the notation of eq. (2.15) for the continued fraction RG, $T_n = R T_1(T_0)^{n-1}$ so that we can recover the continued fraction representation by repeated applications of these fundamental operations.

If $[\xi, \eta]$ has a $p/(p+q) = \{p, q\}$ cycle, the domain of $\xi$ has $q$ elements, and the domain of $\eta$ has $p$ elements. Using this fact, it is easy to show that

$$ p T = \tau p \quad \text{and} \quad p R = \tau p. $$

Thus $p T = \tau p$ and $p R = \tau p$. Note that the symmetry of the winding number about $1/2$ is preserved.

It is easy to show that the space of proper reduced maps is closed under $T$. We have thus reduced the entire renormalization of the circle maps to two symmetric independent operators which shift the binary address of the winding number of the map. In the next section we will explore the set of proper reduced maps which is stable and invariant under this renormalization.

5. Description of the universal invariant set

The previous sections have discussed how the number theory of the Farey tree could be used to construct a renormalization group operator of maps of the circle. This RG operator, acting on the space of reduced maps with irrational winding numbers, can be thought of as a dynamical system whose dynamics eventually approach an invariant set. In this section, we will derive the functional equation that defines the invariant set which we will be able to describe using numerical methods. When the RG dynamics are generated by diffeomorphisms, the invariant set of the dynamical system is the manifold of pure rotations. However, when the RG is applied to non-invertible diffeomorphisms, the resultant invariant set shows more complicated structure.

5.1. Theoretical definition of the invariant set

Let us consider a reduced map defined by eq. (2.10). Using eq. (4.1), we define the $k$th renormalized map by

$$ T_k \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] = T^{k-1} \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]. \quad (5.1) $$

with the index in the composition increasing from right to left. Assume that $\rho([\xi, \eta],_0) = b T[I_0, I_1, \ldots, I_k, \ldots, I_N]$, where $N$ could be infinite. According to eq. (2.20), for large $k$, $[\xi, \eta]_k$ converges to a universal function independent of $[\xi, \eta]_0$ for a fixed $\rho([\xi, \eta],_0)$. This has the following important consequence:

- For large $k$, $[\xi, \eta]_k$ converges to a reduced map which depends only on its winding number and on the history of renormalization.

This can be expressed more precisely by saying that $[\xi, \eta]_k$ converges for large $k$ to a map which
is completely defined by two strings:
\begin{align}
S^k_{\text{head}} &= [I_k, I_{k+1}, \ldots], \\
S^k_{\text{tail}} &= [I_{k-1}, I_{k-2}, \ldots, I_0].
\end{align}
(5.2)

We have chosen to write $S^k_{\text{head}}$ with the $k$th entry first. Thus, the immediate history of the renormalization is indicated by the first entries in $S^k_{\text{head}}$, whereas the immediate future of the RG trajectory is indicated by the first entries in $S^k_{\text{tail}}$.

Let us assume that a reduced map has a winding number $\rho$ which obeys $\rho(\xi, \eta) = b_\xi[I_0, \ldots, I_N]$ and that another map $[\xi, \eta]$ has a winding number $\tilde{\rho}$. Assume that the tails of the address of the winding number of the reduced maps are identical so that the winding number of the two maps become identical after some renormalization steps. From eq. (2.20), if generalized to include proper reduced maps rather than just circle maps, it follows that there exist an $m$ and $\tilde{m}$ such that
\[
\lim_{k \to \infty} T^k T^m \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0.
\]
(5.3)

From this, it is clear that entries on the far right of $S^k_{\text{head}}$ become unimportant for large $k$. Moreover, as long as only entries far to the right of $S^k_{\text{tail}}$ differ, the winding number of the reduced maps will be nearly identical, and hence the binary pattern far to the right in $S^k_{\text{tail}}$ also is not expected to be important at this stage of iteration. We therefore expect that an invariant set to which high RG iterates flow will be a function of the pairs of binary strings $S^k_{\text{head}}$ and $S^k_{\text{tail}}$. We can therefore label the invariant set by $[\xi, \eta]^* (x, S^k_{\text{head}}, S^k_{\text{tail}})$ so that
\[
\begin{bmatrix} \xi \\ \eta \end{bmatrix}_k (x) \overset{k \to \infty}{\longrightarrow} \begin{bmatrix} \xi \\ \eta \end{bmatrix}^* (x, S^k_{\text{head}}, S^k_{\text{tail}}),
\]
(5.4)

where the notation in eqs. (5.1) and (5.2) have been used. Since the far right entries of $S^k_{\text{head}}$ and $S^k_{\text{tail}}$ have decreasing importance, we expect that $[\xi, \eta]^* (x, S^k_{\text{head}}, S^k_{\text{tail}})$ is convergent in the sense of eq. (3.3) as a function of the strings $S^k_{\text{head}}$ and $S^k_{\text{tail}}$.

5.2. Definition of two coordinates, $\omega$ and $\beta$

Since the invariant set can be parametrized by a pair of bi-infinite binary strings and there is a 1:1 correspondence between irrational numbers and bi-infinite binary strings via the binary address, we can simply replace the head string $S^k_{\text{head}}$ and tail string $S^k_{\text{tail}}$ by $\beta_k$ and $\omega_k$ defined by
\[
\beta_k = b_\xi[I_{k-1}, \ldots, I_0], \quad \omega_k = b_\xi[I_k, I_{k+1}, \ldots].
\]
(5.5)

The two coordinates $\beta$ and $\omega$ can be equivalently used to define the invariant set $[\xi, \eta]^* (x, \beta, \omega)$.

According to these definitions, given that $\rho(\xi, \eta) = b_\xi[I_0, \ldots, I_\infty]$, we can write
\[
\begin{bmatrix} \xi \\ \eta \end{bmatrix}_k (x) \overset{k \to \infty}{\longrightarrow} \begin{bmatrix} \xi \\ \eta \end{bmatrix}^* (x, \beta_k, \omega_k).
\]
(5.6)

Although, at this point, $\beta$ and $\omega$ can be thought of simply as shorthand notation for the strings $S^k_{\text{head}}$ and $S^k_{\text{tail}}$, it is a very appealing representation since both the coordinates, $\beta$ and $\omega$, and $[\xi, \eta]^*$ are convergent as a function of the strings.

5.3. RG equations defining the universal invariant set

How do $\beta$ and $\omega$ evolve under renormalization? Clearly $\omega_{k+1}$ is obtained from $\omega_k$ by shifting the address of $\omega_k$ to the left and removing the first binary entry $I_k$. The address of $\beta_{k+1}$ is obtained from $\beta_k$ by adding $I_k$ onto the head of the address of $\beta_k$. The parametrization in terms of $\omega$ and $\beta$ enables us to write $\omega_{k+1} = T^F(\omega_k)$ and $\beta_{k+1} = s^F(\beta_k, \omega_k)$ for some function $s^F$. Like $T^F$ in eq. (3.12) the transformation $s^F(\beta, \omega)$ has two branches $s^F_0$ and $s^F_1$ which add an entry 0 or 1 respectively to the front of the address of $\beta$. We
write this as
\[ s_0^F(\beta) = \frac{\beta}{1 + \beta}, \quad s_1^F(\beta) = \frac{1}{2 - \beta}, \quad (5.7) \]
and
\[ s^F(\beta, \omega) = \begin{cases} s_0^F(\beta) & \text{if } \omega < 1/2, \\ s_1^F(\beta) & \text{if } \omega > 1/2. \end{cases} \quad (5.8) \]

With these definitions, we can write down the universal renormalization equations which the invariant set must obey
\[
T \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^* (x, \beta, \omega) = \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] (x, s^F(\beta, \omega), t^F(\omega)).
\]

(5.9)

The symmetry about 1/2 can be expressed as
\[
R \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^* (x, \beta, \omega) = \left[ \begin{array}{c} \eta \\ \xi \end{array} \right] (x, r(\beta), r(\omega)), \quad (5.10)
\]
where \( r \) is given by eq. (3.14).

Eqs. (5.5), (5.9) and (5.10) are the fundamental renormalization group fixed point equations which must be solved for all values of \( \omega \) and \( \beta \). A special case of this equation occurs for those values of \( \omega \) which yield a cycle under the action of \( t^F \). With our definitions, if \( \omega \) is the golden mean, we obtain a 2-cycle which can be reduced to a 1-cycle using the symmetry relation. Thus, if \( \omega = \sigma_G \) we find
\[
R T \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^* (x, \sigma_G, \omega, \omega) = \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] (x, \sigma_G, \sigma_G). \quad (5.11)
\]

Since \( \sigma_G \) plays as an argument no explicit role in this special case, this is exactly the renormalization equation solved in ref. [2].

5.4. Subcritical manifold

For circle maps with sufficiently small nonlinearity and sufficiently irrational winding numbers, renormalizations eventually flow to the subcritical manifold of pure rotations. The structure of the subcritical manifold \( \{ \xi, \eta \}^*(x, \beta, \omega) \) is simple. The coordinate \( \beta \) becomes irrelevant and the stable manifold is just pure rotations smoothly parametrized by \( \omega \):
\[
\left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^* (x, \omega) = \left[ \begin{array}{c} x + \omega \\ x + \omega - 1 \end{array} \right], \quad (5.12)
\]
and it obeys
\[
T \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] (x, \omega) = \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] (x, t^F(\omega)). \quad (5.13)
\]

It can be verified that this completely solves the subcritical problem.

5.5. Description of the critical set

In order for \( \omega \) and \( \beta \) to be more useful than the equivalent representation using binary strings, these parameters should encode some continuity properties of the invariant set. It is thus natural to ask if \( \omega \) and \( \beta \) can parametrize the critical set without singularities. According to our definition, the parameter \( \omega \) is exactly the winding number \( \rho(f_G) \) as a function of \( \omega \) defines a devil's staircase. Since the parameter \( \Omega \) can be thought of as a one-dimensional cross-section of a \( C^\infty \) manifold of functions, the parametrization as defined by eq. (5.9) should be singular. Questions are: Are there better parametrizations \( \omega \) and \( \beta \) smoothing out the singularities? To what extent are \( \omega \) and \( \beta \) uniquely defined by our prescription?

According to our previous discussions, the fundamental object which defines the critical set is the pair of bi-infinite strings \( S_{\text{head}} \) and \( S_{\text{tail}} \). The conversion of these strings to a pair of real numbers \( \beta \) and \( \omega \), defines a mapping of pairs of bi-infinite binary strings into pairs of real numbers. These mappings implicitly define a pair of shift operators which move the address of the
head of $S_{\text{tail}}$ onto the head of $S_{\text{head}}$. However, the construction of the shift operators can be turned around, and we can take the shift operators $s$ and $t$ as the fundamental objects which implicitly define the coordinates $\beta$ and $\omega$. This is an appealing ansatz, since even well behaved forms for $s$ and $t$ generally induce a singular change of coordinates for $\beta$ and $\omega$.

Assuming that $s$ and $t$ exist, the RG fixed point equations we would like to solve become

$$
T \begin{bmatrix} \xi \\ \eta \end{bmatrix} (x, \beta, \omega) = \begin{bmatrix} \xi \\ \eta \end{bmatrix} (x, s(\beta, \omega), t(\omega)).
$$

(5.14)

Although the equation looks identical to eq. (5.9) here, the only restrictions on $s$ and $t$ are that $s$ and $t$ have some reasonable simple analytic properties and that there exist a pair of idempotent reflection operators $r_s$ and $r_t$. The reflection operators must obey $r_s r_s = \text{Identity}$, $r_t r_t = \text{Identity}$ for which

$$
R \begin{bmatrix} \xi \\ \eta \end{bmatrix} (x, \beta, \omega) = \begin{bmatrix} \xi \\ \eta \end{bmatrix} (x, r_s(\beta), r_t(\omega)).
$$

(5.15)

and $r_s r_s = r_t t_t$. In analogy to eqs. (3.12) and (5.8) we expect that $s$ and $t$ only have two analytic branches. Thus, $s$ and $t$ are to be self-consistently solved for from the RG fixed point equation. If a solution to eq. (5.14) exists for a particular $r$, $s$ and $t$, a new solution can be obtained by a coordinate transformation of $\beta$ and $\omega$. Thus, without loss of generality we can take the parametrization to be symmetric about $\beta = 1/2$ and $\omega = 1/2$, so that $r_s = r_t = 1 - \text{Identity}$.

Then assuming that $s$ and $t$ exist, it is straightforward to determine how they define coordinates $\beta$ and $\omega$ on the invariant set. Assume that $S_{\text{head}} = [I_{k-1}, \ldots, I_0]$ and $S_{\text{tail}} = [I_k, \ldots, I_N]$ are two arbitrary, finite length strings. We first obtain $\beta_k$ by using the recursion

$$
\beta_{k+1} = s_l(\beta_k, \omega_k), \quad \omega_{k+1} = t_l(\omega_k).
$$

(5.16)

Since the symmetry about $1/2$ is assumed, we define $\omega$ and $\beta$ corresponding to the empty string to be $1/2$ as before, so that we demand $\beta_0 = 1/2$ and $\omega_N = 1/2$. We have used the notation $s_l$ and $t_l$ for $l = 0, 1$ to denote the two analytic branches of $s$ and $t$. Provided that each branch of $s$ and $t$ is invertible, this completely determines $1:1$ relation between two coordinates, $\omega$ and $\beta$, and pairs of bi-infinite binary sequences.

The winding number can be simply recovered from the numerical value of $\omega$. We first generate a binary address from $\omega$ by iterating with $t$ until we reach $\omega_N = 1/2$:

$$
1/2 = \omega_N = t_{I_0} t_{I_{N-1}} \cdots t_{I_0}(\omega).
$$

(5.17)

Then, using this binary address, we can compute the winding number by inverting this relation, using $t^{-1}$ instead of $t$. Using the fact that the operator $s^{-1}$ defined in eqs. (5.7) and (5.8) corresponds to the inverse of $t^{-1}$, we write this as

$$
\rho(\omega) = s_{I_0}^{-1} s_{I_1}^{-1} \cdots s_{I_N}^{-1}(1/2).
$$

(5.18)

In general, this prescription leads to a singular reparametrization of $\beta$ and $\omega$. However, it is necessary for smoothing out the parametrization of the critical set.

In the next few sections, we explore the continuity and smoothness properties of this parametrization of the critical set. The results we will present in the next three sections are consistent with Lanford's propositions [6] that under suitable hypotheses there exists a hyperbolic strange set underlying continued-fraction renormalizations of cubic critical proper reduced maps and that a parametrization of the unstable manifolds exists such that the induced action of the RG transformation is $C^1$. 

5.5.1. A smooth coordinate \( \omega \) apparently exists

Independent of the properties of \( s \) and \( t \), a consequence of the existence of a coordinate \( \omega \) which continuously parametrizes the critical set is that every two-dimensional cross-section of the invariant set at fixed \( \beta \) must lie on a one-dimensional smooth curve. We checked this by choosing several fixed head strings of length \( k \), then looking at various cross-sections of a set of \( k \)th renormalized maps whose binary representations of the winding number have identical head strings through length \( k \).

In order to investigate the critical set, we looked at various cross-sections of highly renormalized maps \([\xi_k, \eta_k]\) in eq. (5.1) serving as proxies for the invariant set. In fig. 5(a), we have plotted \( \eta_k'(\xi_k(0)) \) versus \( \xi_k'(\eta_k(0)) \) for \( k = 6 \) with \( \Omega(I_0, \ldots, I_N) \) with \( N = 15 \) and a fixed head string \( S_{\text{head}}^k = [000000] \) for all values of \( I_n \) for \( 6 \leq n \leq 15 \). (We have used the obvious shorthand notation of letting \( \Omega(I_0, I_1, \ldots) \) denote the value of \( \Omega \) such that \( \rho(f) = b f[I_0, I_1, \ldots] \).)

The plot in fig. 5(a) shows a curve with gaps of many different scales which appears to be continuous. However, we must be careful when we discuss functional properties of this curve since it has many gaps where the function is not defined. It has been shown [14] that for a set of all irrational winding numbers this plot produces a Cantor set lying on a one-dimensional curve. The continuity and differentiability are also well defined on this domain of a Cantor set. Assume that a function \( f \) is defined on a domain of a Cantor set, \( D \). Then a function \( f \) is continuous at \( x \in D \) if for every sequence \( \{x_n\}_{n=m}^{\infty} \) obeying \( \lim_{n \to \infty} |x_n - x| = 0 \), \( \lim_{n \to \infty} |f(x_n) - f(x)| = 0 \). A function \( f \) is differentiable with a derivative \( f'(x) \) at \( x \in D \) if for every sequence \( \{x_n\}_{n=m}^{\infty} \) obeying \( \lim_{n \to \infty} |x_n - x| = 0 \), \( f'(x) = \lim_{n \to \infty} [f(x_n) - f(x)] / (x_n - x) \) exists. Another point of view is that if we define a suitable function \( f \) on a complement of \( D \) which smoothly interpolates the regions of gaps in the curve, the region where the functional properties of \( f \) hold can be extended from the domain \( D \) to the entire interval.

The curve in fig. 5(a) can be said to be continuous in either sense. A detail of fig. 5(a) shown in fig. 5(b) confirms the continuity. By computing the numerical finite difference, the differentiability of the curve can be seen by the plot of one of its coordinate and the slope of the previous curve as shown in fig. 6. The first derivative appears to exist and is continuous up to the precision of our numerical data. Numerical
evidence suggests that the second derivative also exists and is continuous.

Another cross-section of the critical set is shown in fig. 7 by plotting
\[ \eta_k(\xi_k(0))(\Omega(I_0, \ldots, I_k, \ldots, I_N)) \]
versus \( \Omega(I_0, \ldots, I_N) \) for all possible values of the tail string \( S^\text{tail}_k = [I_k, \ldots, I_N] \) and a fixed head string \( S^\text{head}_k = [010100] \) with \( k = 5, N = 11 \). The curve appears to be continuous and differentiable. Again, numerical evidence suggests that the coordinate is at least twice continuously differentiable. Similar experiments were done for other values of \( \beta \). All results are consistent with the fact that for each \( \beta \), a valid coordinate \( \omega \) exists which is at least \( C^2 \) smooth. Thus, the assumption that the parameter \( \omega \) defining a manifold exists is consistent with the numerical data. This result indeed is consistent with the Lanford's result that under suitable hypotheses there exists the \( C^1 \) parametrization of the \( \omega \) coordinate [6].

5.5.2. The coordinate \( \beta \) is not continuous

We will now discuss whether or not a coordinate \( \beta \) could exist which continuously parametrizes the irrelevant direction on the invariant set. The numerical calculation is similar to that in section 5.5.1, but now we fix the tail string of the winding number, and look at all possible winding numbers with different head strings of length \( k \) keeping the tail strings identical. First, we chose the coordinate

versus \( \Omega(I_k, \ldots, I_N) \) with \( k = 5 \) and \( N = 11 \) for a fixed head string \( S^\text{head}_k = [010100] \), and all values of \( I_n \) for \( 5 \leq n \leq 11 \). The curve appears to be continuous and differentiable. Again, numerical evidence suggests that the coordinate is at least twice continuously differentiable. Similar experiments were done for other values of \( \beta \). All results are consistent with the fact that for each \( \beta \), a valid coordinate \( \omega \) exists which is at least \( C^2 \) smooth. Thus, the assumption that the parameter \( \omega \) defining a manifold exists is consistent with the numerical data. This result indeed is consistent with the Lanford's result that under suitable hypotheses there exists the \( C^1 \) parametrization of the \( \omega \) coordinate [6].

Fig. 6. The derivative of the curve in fig. 5(a) is shown. It appears to exist and is continuous. This illustrates the one-dimensional character of the critical set in the variable \( \omega \).

Fig. 7. A plot of \( \eta(\xi(0))(\omega) \) versus \( \omega \) at fixed \( \beta \) is shown. The plot was obtained by plotting
\[ \eta_k(\xi_k(0))(\Omega(\rho(I_0, \ldots, I_k, \ldots, I_N))) \]
versus \( \Omega(I_0, \ldots, I_N) \) for all possible values of the tail string \( S^\text{tail}_k = [I_k, \ldots, I_N] \) and a fixed head string \( S^\text{head}_k = [010100] \) with \( k = 5, N = 11 \). The curve appears to be continuous and the smoothness of the curve is consistent with its being differentiable.

Fig. 8. A plot of \( \eta(\xi(0))(\beta) \) at fixed \( \omega \) is shown. This is obtained by plotting \( b_\rho(I_k, \ldots, I_N) \) versus \( \eta_k(\xi_k(0))(\Omega(\rho(I_0, \ldots, I_k, I_N))) \) for all possible values of the head string \( S^\text{head}_k = [I_k, \ldots, I_N] \) and a fixed tail string \( S^\text{tail}_k = [I_k, \ldots, I_N] = [01010000] \) with \( k = 8, N = 14 \). The curve is obviously a highly singular function of this coordinate.
and plotted $\beta$ versus $\eta_k(\xi_k(0))$ with $k = 8$ and $N = 14$, for a tail string $S_{\text{tail}}^k = [0101000]$, and all possible values of the head string of length 8. The result of this is shown in fig. 8. The curve is clearly a highly singular function. In order to check whether this could be caused by our special choice of $\beta$, we plot another cross-section $\xi_k(\eta_k(0))$ versus $\eta_k(\xi_k(0))$ with $k = 10$ and $N = 12$, for a fixed tail string $S_{\text{tail}}^k = [000]$ and for all possible values of the head string of length 10. (See fig. 9(a).) Details of the lower branch are shown successively in fig. 9(b) and fig. 9(c) and confirm the existence of infinite leaves on finer scales.

5.5.3. How many smooth coordinates are needed to embed the critical set?

Numerical results in the previous sections show that the critical set cannot be embedded in a simple smooth two-dimensional geometry. Then we may ask how many dimensions are needed to embed the critical set? That is, how many smooth coordinates are needed for $\beta$? To answer this question, we considered $d$-dimensional cross-sectional data analogous to fig. 9(a) for which we computed the correlation exponent $\nu$ [26] given by

$$C(l) = \sum_{i,j} P(x_i, y_j, l) \lim_{l \to 0} l^n,$$

where $P(x, y, l) = 1$ if $|x - y| < l$ and zero otherwise. The $x_j$ represent the $d$-dimensional data obtained by computing a representative $d$-dimensional cross-section of the critical set for various values of $d$. We have chosen an ensemble of cross-sections, $\eta(0)$, $\xi(0)$, $\eta'(\xi(0))$, $\eta'(\xi(0)/2)$, $\eta'(\xi(0)/3)$, $\xi'(\eta(0)/2)$, $\xi'(\eta(0)/3)$, $\eta'(\xi(0)/4)$, and $\xi'(\eta(0)/4)$. As $d$ is increased, the value where the correlation dimension stabilizes determines an embedding dimension of the critical set.

Though the determination of the saturating dimension is inconclusive, this calculation shows that $\nu$ is at least larger than 1.3. Therefore we
need more than one coordinate to represent the $\beta$ coordinate. Since it is shown [14] that the cross-section of a critical set for a fixed $\beta$ has a dimension $D_\omega \approx 0.87$, the total dimension of the critical set is larger than $\nu + D_\omega \approx 2.2$. Therefore we conclude that the embedding dimension of the critical set is at least larger than two. A consequence of this would be that the fixed point equation in eq. (5.14) for the function $s$, $t$ and $[\xi, \eta](x, \beta, \omega)$ are impossible to solve as long as we insist that $\beta$ and $\omega$ determine parameter values on the invariant set and $t$ is analytic.

Though the critical set does not admit the smooth two-parameter geometry, we have at least shown that the cross-section of the critical set at fixed $\beta$ lies on a one-dimensional smooth curve. We can therefore choose some suitable one-dimensional parametrization for the manifold at each value of $\beta$ to parametrize the invariant set. The dynamics on the invariant set can therefore be written as $T[\xi, \eta](x, \beta, \omega) = [\xi, \eta] [x, s(\beta, \omega), t_\beta(\omega)]$. This is essentially the same as eq. (5.14), but no continuity properties have been assumed for $s(\beta, \omega)$ or $t_\beta$. We now ask an important question: recognizing that $\omega$ can be chosen which provides a $C^2$ parametrization, can we choose $\omega$ globally so that $t_\beta$ does not depend on $\beta$? If this were true, it would have enormous consequences since $t(\omega) = t_\beta(\omega)$ would then encode all important scaling properties of the devil's staircase.

A consequence of the assumption that $t_\beta$ is independent of $\beta$ is that the coordinate $\omega$ can be chosen consistently for all $\beta$ and that for two distinct choices of $\beta$, $[\xi, \eta](x, \beta_1, \omega)$ is a differentiable function of $[\xi, \eta](x, \beta_2, \omega)$. To check this, we plot the renormalized $\eta_\beta(0, \beta_1, \omega)$ versus $\eta_\beta(0, \beta_2, \omega)$, both parametrized by $\omega$ corresponding to all possible values of tail strings of length 9. The data is shown in fig. 10 with $k = 5$ for fixed head strings, $S^\text{head}_x = [01010]$ and [10100]. Clearly the curve is continuous. The slope of this curve shown in fig. 11 suggests that the derivative exists, whereas higher derivatives appear to be undefined. We therefore conclude that we can choose a consistent parameterization of the unstable coordinate of the invariant set so that, up to a $C^1$ change of coordinates, single function $t(\omega)$ determines the renormalization of the coordinate $\omega$. In section 7 we explore the analytic properties of $t(\omega)$ and the consequences for universal scaling.
6. Analytic structure of the devil's staircase

The present section will be devoted to exploring the analytic structure seen in the devil's staircase. In section 7, we will relate this to universal properties of the invariant set which we discussed in the previous section.

A universal feature, by definition, is model independent. In order to understand what features are universal for circle maps, we will compare the devil's staircase for the standard sine map \( f \) in eq. (1.3) with another for an arbitrarily chosen circle map \( g \) given by

\[
g(x) = x + \Omega - \frac{5a}{16\pi} \left( \sin(2\pi x) + \frac{1}{3} \sin(6\pi x) \right).
\]  

(6.1)

We call \( \Omega \) a generic parametrization of the circle map if \( f_\Omega \) is a \( C^\infty \) parametrization of circle maps and \( \rho(f_\Omega) \) is monotone. In what follows, we will assume that \( \Omega \) is always a generic parametrization.

A circle map is called cubic if it is a generic circle map with exactly one cubic inflection point at the origin. The above two maps at \( a = 1 \) are cubic since they develop cubic singularities at the origin. These cubic maps display the same universal features [2].

In order to define exactly what we mean by universal, it is necessary to make some definitions. The function \( \rho_f(\Omega) \) is defined in the obvious way: \( \rho_f(\Omega) = x \) if \( \rho(f_\Omega) = x \). The inverse function \( \Omega_f \) is defined similarly: \( \Omega_f(y) = \Omega \) if \( \rho(f_\Omega) = y \), so that \( \rho_f(\Omega_f(x)) = x \). Thus, the devil's staircase in fig. 1 is simply a plot of \( \rho_f(\Omega) \). A plot of \( \rho_g(\Omega) \) gives another devil's staircase for the map \( g \). (See fig. 12.) These devil's staircases are generated by computing the parameter values of \( \Omega \) which place the origin on a superstable cycle [22] for all winding numbers of the Farey tree through level 12. Clearly these two staircases look very similar. We define a function \( H_{fg} \) relating these devil's staircases by

\[
H_{fg}(\Omega) = \Omega_g(\rho_f(\Omega)),
\]  

(6.2)

where the range of \( H_{fg} \) is the range of \( \Omega_g \), and the domain of \( H_{fg} \) is the range of \( \Omega_f \).

Fig. 12. A devil's staircase for a map, \( g(x) = x + \Omega_g - (5/16\pi)(\sin(2\pi x) + 0.2\sin(6\pi x)) \), It is a plot of \( \rho(g,\Omega_g) \) versus \( \Omega_g \), essentially similar to one for a map \( f \) in fig. 1.

For the description of the functions \( H_{fg} \), \( \rho_f \), and \( \Omega_f \), we need some notions of continuity and smoothness beyond the usual categories of \( C^n \). A function \( f: D \to \mathbb{R}^{-1} \) is Lipschitz continuous (or Lipschitz) on its domain \( D \) if there exists a finite \( C > 0 \) such that \( |f(x) - f(y)| \leq C|x - y| \) for all \( x, y \in D \). In the following discussions, \( D \) is taken to be a subset of points in an interval \([0, 1]\).

Several simple statements can be made about Lipschitz continuity:

- If \( f \) and \( g \) are Lipschitz, then, provided composition is well defined, \( fg \) is also Lipschitz.
- If \( f^{-1} \) exists and is Lipschitz, there is a constant \( C \) so that \( C|x - y| \leq |f(x) - f(y)| \).
- If \( f \) is Lipschitz, it does not imply that \( f^{-1} \) is Lipschitz.

For future analysis, let \( \Omega_f^\mathbb{R}(\rho) \) denote the function \( \Omega_f \) for the domain \( D \) consisting of all irrational values of \( \rho \), and \( \Omega_f^{\mathbb{R}}(\rho) \) denote the analogous function for all rational values of \( \rho \). When \( \rho \) is rational, \( \Omega_f \) is constant over a finite interval of \( \Omega \). To make the function \( \Omega_f \) well defined, we define it to be the unique value of \( \Omega \) which places the
origin on a superstable cycle. We also define $H_{fg}$ and $H_{sg}^{irr}$ to be the restrictions of $H_{fg}$ on ranges of $Q_{fg}^{irr}$ and $Q_{sg}^{rat}$, respectively.

Based on the notion of Lipschitz continuity we state several properties of the function $p$, which are generic to cubic circle maps.

**Conjecture 1.** $p_f$ is not Lipschitz.

We plot $p$ vs. log $\Delta \Omega / \Delta \rho$, where $\Delta$ refers to the finite difference from two $\rho$’s which are the numerically closest neighbors in level 12 Farey tree and above. (See fig. 13.) In order to prove conjecture 1, it is sufficient to show that $\Delta \Omega / \Delta \rho$ approaches zero for some winding numbers. Repetitive patterns seen in fig. 13 indicate the convergence of data points to a small number as the binary address of the winding number gets longer. We know that the slope $\delta_{rat}$ is bounded below by zero since $\delta$ is monotonically increasing in $\rho$. Computation for a set of irrational winding numbers confirms that the slope is indeed converging to zero.

Conjecture 1 is, in fact, a simple consequence of the scaling theory in ref. [2]. A dense set of irrationals has a golden-mean sequence of alternating 0’s and 1’s in the tails of their binary address. Assume that $(\rho_n)_{n=0}^\infty$ is a sequence of rational approximants to a given irrational number with golden-mean tail sequences and $\rho(f_{sg}^{rat}) = \rho_n$. Then a sequence of parameter values $Q_{sg}^{rat}$ are known to obey the following scaling relations [1, 2, 27]:

$$\lim_{n \to \infty} \frac{\Delta Q_{sg}^{rat}}{\Delta \rho_{n-1}} = \delta_{crit} \quad \text{and} \quad \lim_{n \to \infty} \frac{\Delta \rho_n}{\Delta \rho_{n-1}} = \delta_{sub}$$

where $\delta_{crit} = 2.83360$ and $\delta_{sub} = 2.61803$. It follows that

$$\lim_{n \to \infty} \frac{\Delta Q_{sg}^{rat}}{\Delta \rho_n} = \delta_{crit} = 1.08234.$$  \hspace{1cm} (6.3)

Therefore, $\Delta Q_{sg}^{rat} / \Delta \rho_n$ converges geometrically to zero in the $n \to \infty$ limit, and, at least for these infinite numbers of points, $\rho_f(\Omega)$ is not Lipschitz.

We now turn to the analytic properties of $H_{fg}$. David Rand has made conjectures about this function [9]:

**Conjecture 2.** (due to David Rand). $H_{fg}^{rat}: Q_{fg}^{rat} \to Q_{sg}^{rat}$ is Lipschitz if $f$ and $g$ are two cubic circle maps.

Generically, $H_{fg}^{rat}$ is also Lipschitz since generic properties do not distinguish the functions $f$ and $g$. Since $H_{fg}^{rat}$ is the functional inverse to $H_{fg}$, both functions possess Lipschitz inverses.

Fig. 14 shows $H_{fg}^{rat}$ which is a plot of $Q_{fg}^{rat}$ versus $Q_{sg}^{rat}$. We now plot the $\Omega_f$ versus $\Delta Q_{sg}^{rat} / \Delta \Omega_f$ for winding numbers corresponding to all Farey elements through level 12 in fig. 15. $\Delta Q_{sg}^{rat} / \Delta \Omega_f$ is computed by the finite difference method discussed before. From fig. 15, it can be seen that ratios are bounded above and below away from 0. One may be concerned about the seemingly diverging wings of clusters at either edge of gaps; these data points can be shown to correspond to winding numbers with tails of all 0’s or all 1’s in their binary address. The numerical calculation of $\Delta Q_{sg}^{rat} / \Delta \Omega_f$ for these values of $\Omega$ approaching either edge of some systematically chosen gaps shows that the ratios are converging to finite nonuniversal constants. This numerical evidence supports the assertion that $H_{fg}^{rat}$ and its inverse are Lipschitz. Taking the closure of the domain and
range of $H_{fg}$, and restricting this set to the collection of limit points would result in Lipschitz continuity of $H_{fg}$.

In the case of periodic irrationals, conjecture 2 can be understood by using well-known scaling arguments. For an irrational number with periodic binary address, a fixed point of the renormalization group operator exists which is a product of a sequence of the binary renormalization operator in eqs. (4.2) and (4.3). Let us assume that the largest unstable eigenvalue of the linearized renormalization operator is $\delta (>1)$ as defined in eq. (2.7). It is well established [2] that two generic critical maps, say $f$ and $g$, flow to the same fixed point. Thus the same exponent $\delta$ describes the scaling of $\Omega_f$ and $\Omega_g$, so that $(\Delta \Omega_f)_n = C_f \delta^{-n}$, $(\Delta \Omega_g)_n = C_g \delta^{-n}$, where $C_f$ and $C_g$ are nonzero finite constants. Therefore $(\Delta \Omega_g / \Delta \Omega_f)_n \to n \to \infty C_g / C_f$.

On the basis of numerical data, we have strengthened conjecture 2 to include differentiability. Though the domain $D$ of $H_{fg}$ is not an entire interval, we can still define a differentiability on a domain $D$ as discussed in section 5.5.1. Note that bounded differentiability on $D$ implies Lipschitz continuity. We then assert

**Conjecture 3.** $H_{fg}: \Omega_f \to \Omega_g$ is differentiable if $f$ and $g$ are two generic cubic circle maps. By genericity arguments, it follows that $H_{fg}$ is also differentiable.

From fig. 15, it can be seen that for each gap corresponding to a rational winding number there exist distinct left and right limit slopes $\Delta \Omega_g / \Delta \Omega_f$ for winding numbers approaching either edge of the gap (the tips of seemingly diverging wings). The set of these limit slopes at the edges of gaps can define an envelope function containing all data points in fig. 15, so that this function would provide the upper and lower bounds for a cluster of data points $\Delta \Omega_g / \Delta \Omega_f$ between any pair of gaps. Inspection of fig. 15 indicates the strong correlation between the sizes of the domain and range of the cluster between any pair of gaps. It can be numerically checked that for a set of systematically chosen irrational winding numbers the range of $\Delta \Omega_g / \Delta \Omega_f$ shrinks to zero as the size of its domain containing a given irrational decreases, which indicates that $H_{fg}$ is differentiable. Differentiability of $H_{fg}$ is consistent with the results for $t(\omega)$ which will be discussed in the next section.
7. Scaling and the devil's staircase

This section will tie together section 5 and section 6. We will show that the properties of $H_{f_g}$ described in section 6 and the existence of $t(\omega)$ is consistent, and how $t(\omega)$ determines all universal information about scaling of small gaps in the devil's staircase.

7.1. Scaling function $t(\omega)$ and scaling

As before, the relation between $\omega$ and its address is denoted by

$$\omega = \omega(I_0, I_1, \ldots, I_N).$$

(7.1)

Assume that $\omega = \omega(I_0, \ldots, I_k, I_{k+1}, \ldots, I_N)$ and $\tilde{\omega} = \omega(I_0, \ldots, I_k, \tilde{I}_{k+1}, \ldots, \tilde{I}_N)$ differ by at least one element in the address. Using the definition of the scaling function $t$ in eq. (5.14), we can write

$$t(\omega(I_0, \ldots, I_k, I_{k+1}, \ldots, I_N)) = \omega(I_1, \ldots, I_k, I_{k+1}, \ldots, I_N),$$

(7.2)

and

$$t(\omega(I_0, \ldots, I_k, \tilde{I}_{k+1}, \ldots, I_N)) = \omega(I_1, \ldots, I_k, \tilde{I}_{k+1}, \ldots, \tilde{I}_N).$$

(7.3)

Expanding the difference in eqs. (7.2) and (7.3) in terms of the derivative of $t$, we find the scaling result

$$\lim_{k \to \infty} \frac{d}{d\omega} t(\omega(I_0, \ldots, I_N)) = \delta(\omega(I_0, \ldots, I_N)), \quad \omega = \omega(I_1, \ldots, I_k, I_{k+1}, \ldots, I_N).$$

(7.4)

All evidence indicates that commuting reduced maps of the circle flow to the same universal invariant set, $[\xi, \eta]^*(x, \beta, \omega)$. Accordingly, if fixed $\beta$, the coordinate $\omega$ serves as a generic parametrization of a critical set. Then, according to the conjectures in section 6, up to a differentiable change of the coordinates we can discover $t(\omega)$ simply by plotting parameter values of $\Omega$ for the standard sine map using

$$t(\Omega(I_0, I_1, \ldots, I_N)) = \Omega(I_1, \ldots, I_N).$$

(7.5)

In the $k \to \infty$ limit the derivative can be obtained by

$$\frac{d}{d\Omega} t_g(\Omega(I_0, I_1, \ldots, I_N)) = \left[ \frac{\Omega(I_1, \ldots, I_k, I_{k+1}, \ldots, I_N)}{\Omega(I_0, \ldots, I_k, \tilde{I}_{k+1}, \ldots, \tilde{I}_N)} \right] \left[ \frac{\Omega(I_0, \ldots, I_k, I_{k+1}, \ldots, I_N)}{\Omega(I_0, \ldots, I_k, \tilde{I}_{k+1}, \ldots, \tilde{I}_N)} \right].$$

(7.6)

for any choice of $N, \tilde{N}, (I_j, \tilde{I}_j)_{j \leq k}$. In the $N \to \infty$ limit, the derivatives of $t_g(\Omega_g)$ and $t_f(\Omega_f)$ are related by

$$\frac{d}{d\Omega} t_g(\Omega(I_0, I_1, \ldots, I_N)) = \left[ \frac{d}{d\Omega} t_f(\Omega(I_0, I_1, \ldots, I_N)) \right] \frac{d}{d\Omega} H_{f_g}(\Omega(I_0, I_1, \ldots, I_N)),$$

(7.7)

The derivative of $t(\Omega)$ is not a universal quantity since the derivative of $H_{f_g}$ is not, in general, a constant function. However, for a quadratic irrational winding number with a periodic tail sequence of period $L$, a consequence of the product rule of derivatives in eq. (7.7) and the fact that repeated applications of $t(\omega)$ yield a cycle implies that the product of derivatives $\prod_{j=0}^{L-1} d\Omega(I_j, I_{j+1}, \ldots, I_N)/d\Omega$ is universal and is identical to $\delta(\rho)$ defined in eq. (2.7).

For an arbitrary irrational winding number, the product of $L$ derivatives defined above is identical for two critical circle maps except a finite multipli-
cation factor if both maps belong to the same universality class of cubic critical circle maps. In the limit \( L \to \infty \) the \( L \)th root of this product of derivatives is well defined and universal. Furthermore the continuity and differentiability of \( d\Omega(t)/d\Omega \), if proved, will be carried over to \( dt(\omega)/d\omega \). Thus \( t(\Omega) \) provides a good representation of \( t(\omega) \).

The function of Lyapunov exponents \( t(\omega) \) is, in general, a function of \( \beta \). However, the coordinate \( \beta \) represents the direction in the set of functions which is attracted onto the invariant set. Thus the coordinate \( \beta \) does not affect the scaling features provided they do not couple to \( t(\omega) \).

### 7.2. Subcritical scaling

It is shown before that the subcritical manifold is the manifold of pure rotations. From standard renormalization group arguments we then know that the parameter axis \( \Omega \) of the circle map obeys the same scaling relations as pure rotations at parameter values which give high order cycles or irrational values of the winding number. For the subcritical problem, the scaling function \( t \) is \( \mathcal{C}^\infty \) and can be shown to be equivalent to \( t^F \). The function of Lyapunov exponents for arbitrary winding number \( \rho \) is given by

\[
\delta(\rho) = \begin{cases} 
(1 - \rho)^{-2}, & \rho < 1/2, \\
\rho^{-2}, & \rho > 1/2.
\end{cases} \tag{7.8}
\]

### 7.3. Critical scaling

The scaling function \( t(\omega) \) for critical scaling is shown in fig. 16(a) by plotting \( \Omega(I_0, \ldots, I_N) \) versus \( \Omega(I_1, \ldots, I_N) \) for the standard sine map \( f \). The data is generated from all addresses of length less than or equal to 12. A detail in fig. 16(b) shows the vicinity of the point \( 1 - \alpha \). The curve appears to be continuous. In contrast to the subcritical problem which lacks a stable fixed point, the critical scaling function \( t(\omega) = \omega \) has a solution, so that there will be an infinity of gaps generated provided \( t_0 \) or \( t_1 \) maps a point into the attracting domain of the fixed point. Then \( \omega \) gets *trapped* by the fixed point under successive recursions. The slope of \( t(\omega) \) computed numerically by using eq. (7.6) is shown in fig. 17(a). Note that the derivatives exist for irrational values of \( \omega \), though the right and left limits of the slope are different at either edge of gaps. The derivative appears to be stable against increasing the length of the string.
used in the finite difference. We cite this as further evidence that \( t(\omega) \) is differentiable. Though \( t(\omega) \) is continuous and differentiable, it can be seen from the blow up of fig. 17(a) around \( 1 - \alpha_3 \) in fig. 17(b) that the \( t(\omega) \) is not twice differentiable.

7.4. Approximation of the devil’s staircase

Though \( t(\omega) \) is not \( C^\infty \), it is possible to crudely approximate \( t(\omega) \) by a \( C^\infty \) function and, therefore, the devil’s staircase. Numerical results of scaling near large gaps for the standard sine map indicate that

\[
\frac{dt(\omega)}{d\omega}
\]

when \( t(x_*) = x_* \),

\[
\frac{d^2t(\omega)}{d\omega^2}
\]

(7.9)

The second derivative equation is not expected to be universal, but is accurate for the sine map. It can be obtained numerically by looking at the convergence of the exponent \( \delta \) defined by eq. (7.4) to its asymptotic value of 1 for the sequence of winding numbers of \( \rho_n = 1/n \). We also know how the exponents in table I must be related to derivatives of \( t \). For instance,

\[
-\delta(\sigma_0) = \frac{dt_0(\omega)}{d\omega}
\]

(7.10)

The following simple function provides an excellent numerical approximation to \( t(\omega) \):

\[
t(\omega) = \frac{0.02333 + 0.65925\omega}{1 - 1.24440\omega}.
\]

(7.11)

It was obtained by fitting the form to eqs. (7.9) and (7.10), where \( \delta \) was obtained from table I. This function has all the features necessary to generate a devil’s staircase of the type seen in the plot of \( \Omega \) vs. \( \rho \). Using this approximation to \( t(\omega) \) and eq. (5.17) we can generate a mock devil’s staircase. This is shown in fig. 18 by plotting the mock \( \omega \) versus \( \rho \). This staircase is very similar to one in fig. 1 containing all phase locked tongues of approximately right scales. In fig. 19 we plot \( \omega \) for a standard sine map at criticality versus the mock \( \omega \).

This function looks quite smooth, and the scaling of the gaps on the fine scale is reproduced very...
Summary and discussion

We believe the RG presented in this section is conceptually simpler and more elegant than the continued fraction formulation. Most previous renormalization approaches to the scaling in circle maps has focused on a measure zero set of special winding numbers called quadratic irrationals. This renormalization group is designed to extend the results of the continued fraction RG to arbitrary winding numbers focusing on the global structure of the critical set. The single universal scaling function, which is essentially the set of Lyapunov exponents $\delta$, can be constructed to describe scaling of all small gaps in the devil's staircase for all cubic critical maps. Though we were able to make a simple approximation of the devil's staircase, we could not construct a systematically convergent approximation because of the non-smoothness of the scaling function. We have not been able to find the smooth parametrization of the critical set which would be essential to the systematic functional analysis of the critical set. Rigorous work by Lanford, as well as the numerical work presented here and elsewhere suggests that this goal may not be attainable.

Overall, the picture we present supports Lanford's conjecture of the existence of the hyperbolic strange set underlying continued-fraction renormalizations of cubic critical proper reduced maps under suitable hypotheses [6]. Our description of the smooth unstable manifold of the criti-
cal set is also consistent with Lanford's proposition that under suitable hypotheses a parametrization of the unstable manifolds exists such that the induced action of the renormalization operator is $C^1$ on them.

Farmer, Satija and Umberger [7, 8] have also computed properties of the critical set. Their approach was based on the continued fraction renormalization and they did not consider a systematic parametrization of the critical set. Instead, they explored both a specific random seed winding number and a Monte Carlo ensemble of random winding numbers to obtain only “average” scaling exponents associated with the critical set, while we get a function of Lyapunov exponents.

Similar renormalizations can be constructed for other problems, though the space on which the renormalization operator acts are different. The present renormalization method can be applied directly to one-dimensional Schrödinger equation with a quasi-periodic step-function potential for arbitrary incommensurability to study the scaling properties of the energy spectrum [17]. Another relevant problem is the breakup of any KAM torus with arbitrary winding number in Hamiltonian systems [28]. Mackay et al. made a similar approach to ours to study the universal small scale structures near the boundary of Siegel disks of arbitrary winding number [5]. It was shown that similar strange sets underly the renormalizations in these problems, which recently led Rand to propose a systematic, universal approach to the problem of global scaling associated with these strange sets [9]. It is our hope that this work would contribute to the formulation of the universal theory for scaling in general incommensurate systems.

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References

[3] A critical invariant set can be viewed as an attractor in the sense that all nearby maps restricted to the proper space of critical circle maps should converge to this set under RG transformations.
[4] Robert Mackay's idea has been privately communicated to us by B. Shraiman (1984).
[12] In this context, most irrationals refer to irrationals poorly approximated by rationals, i.e. for every $p, q$ there is a constant $C$ and exponent $v$ such that $|\sigma - p/q| > C/q^{2+v}$. This set is of full measure in $\mathbb{R}^1$.
[21] Composition of operators $f$ and $g$ or functions $f$ and $g$ is always implied by $fg$.
[22] The derivative of $q$ times iterated map at a superstable cycle $p/q$ is zero; therefore a superstable cycle is maximally stable. For a standard sine map at criticality, the origin always belongs to a superstable cycle.
[25] Refs. [18] and [19] used a slightly different binary labeling scheme. There exist a simple one to one correspondence between our binary scheme and theirs.