Mini-Workshop on Exactly Solvable Models and Their Applications to Cold-Atom Systems
Chinese University of Hong Kong
June 20 – 22, 2008

Entanglement of Fermions at Quantum Criticality: Exact Results

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VETENSKAPSRÅDET

supported by the
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Outline

Entanglement: some basics

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Entanglement and quantum phase transitions (QPTs)

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Fermionic entanglement at QPTs

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Entanglement: some basics

"...best possible knowledge of a whole does not necessarily include the same for its parts. [...] The whole is in a definite state, the parts taken individually are not. [This is] not one, but the essential trait of the new theory, the one which forces a complete departure from all classical concepts." *Schrödinger, 1935*

Entanglement: some basics

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Einstein, Podolsky, Rosen 1935

non-local quantum correlations

"...spooky action at a distance" (Einstein)

violation of "local realism" (*Bell's inequalities*)
verified experimentally (Aspect *et al.* 1982)

Entanglement is a *physical resource*

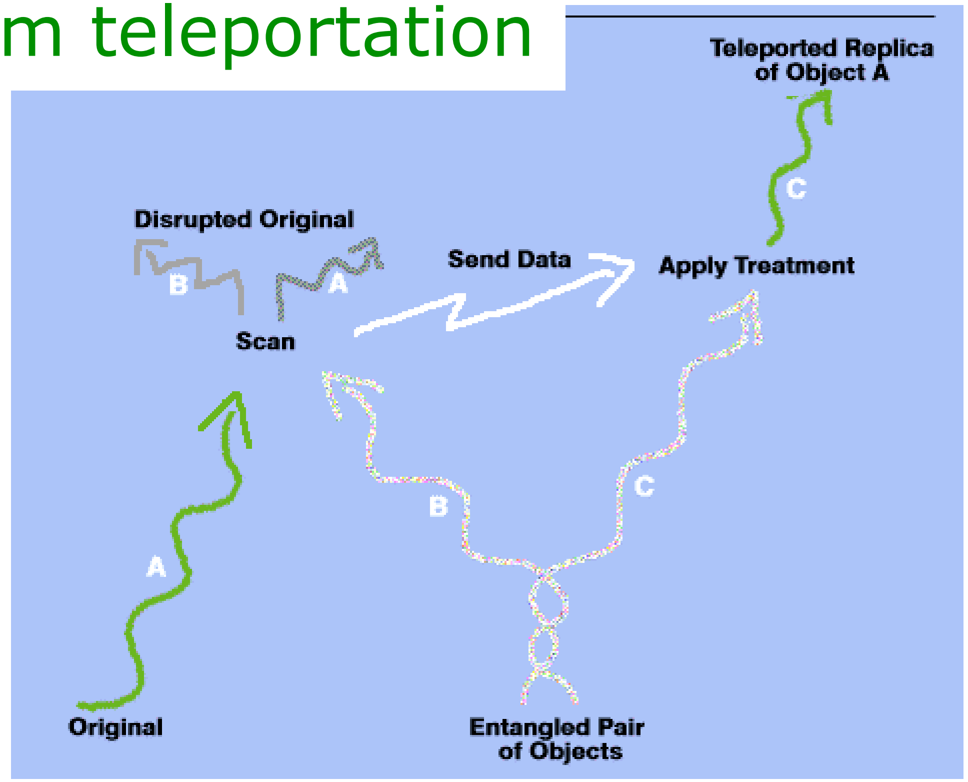
Quantum cryptography

Quantum mechanicians at work:
Installing an optical quantum channel
at the Austrian-Croatian border, 2005

Entanglement is a *physical resource*

Quantum cryptography

Quantum teleportation



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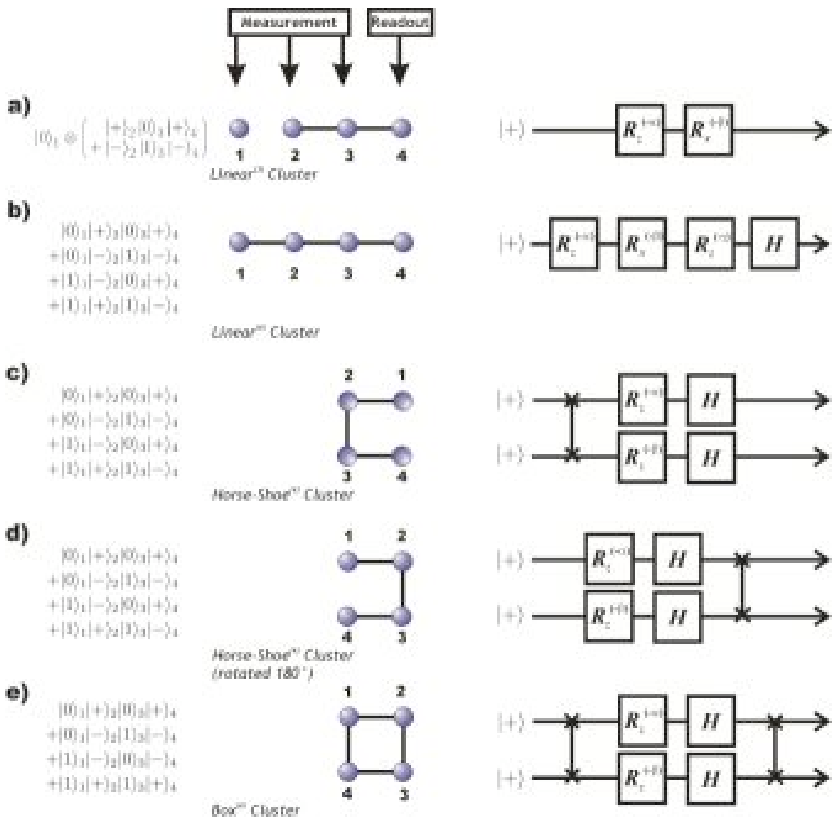

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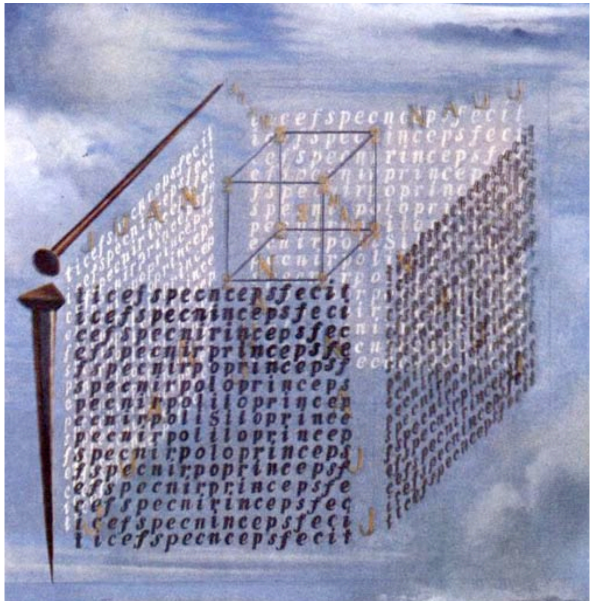
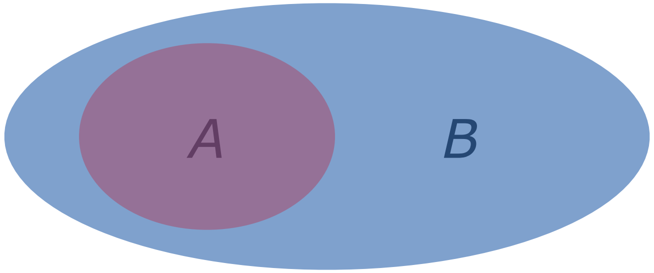
Future technologies?



S. Dali, "Linear Cube"

If entanglement is a *resource,* how to quantify it?



quantum system in a **pure state** $|\Psi(A,B)\rangle$

How "much" entanglement $\mathcal{E}$ between *A* and *B*?

Bennett et al., PRA 53, 2046 (1996):

$\mathcal{E} = -\mathrm{Tr}(\rho_A \log \rho_A)$ *"von Neumann entropy"*

$\rho_A = \mathrm{Tr}_B \rho = \mathrm{Tr}_B |\Psi(A,B)\rangle\langle\Psi(A,B)|$

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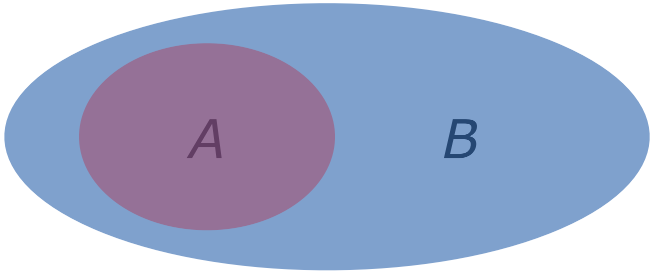

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unique measure of (bipartite) entanglement in a *pure* state

- *non-increasing under local transformations*
- *vanishing for separable states*
- *additive*

Entanglement: some basics

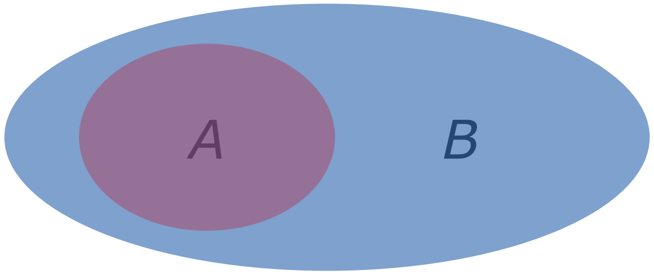
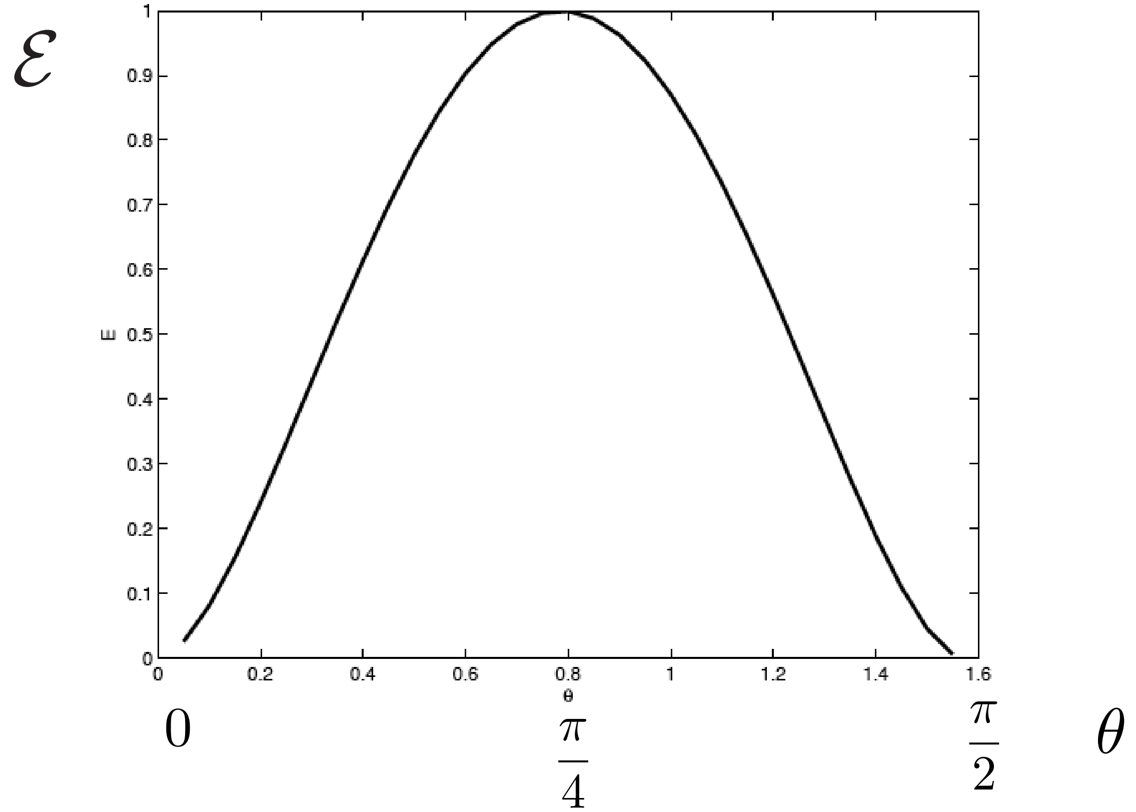


Illustration:

$$|\Psi(A,B)\rangle = \cos\theta\,|\uparrow\rangle_A|\downarrow\rangle_B + \sin\theta\,|\downarrow\rangle_A|\uparrow\rangle_B$$



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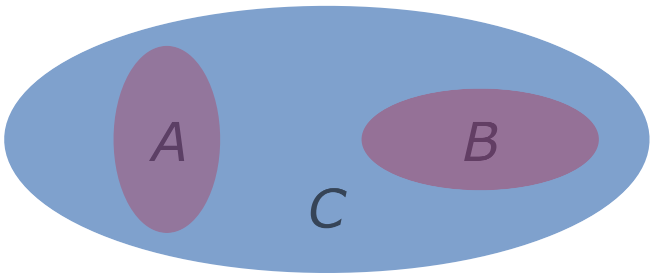


A+B in a **mixed state** $\rho_{AB} = \sum_i p_i \, |\psi_i(A,B)\rangle\langle\psi_i(A,B)|$

(after tracing out C)

How "much" entanglement between *A* and *B*?

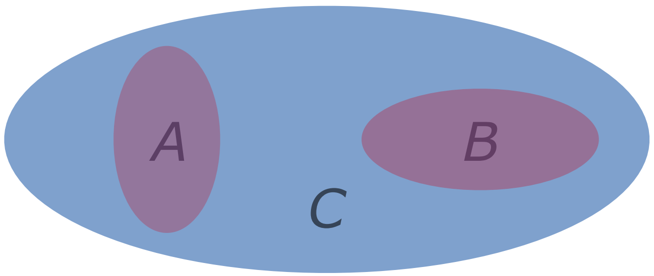
computable measure for ***qubit*** *("two-level") systems*: CONCURRENCE

$$\mathrm{C} = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$ ordered eigenvalues to $\rho_{AB} \times (\rho_{AB})^*$

Wootters, PRL 80, 2245 (1998)

Entanglement: some basics



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How "much" entanglement between *A* and *B*?

encodes the "Entanglement of formation"

computable measure for ***qubit*** *("two-level") systems*: CONCURRENCE

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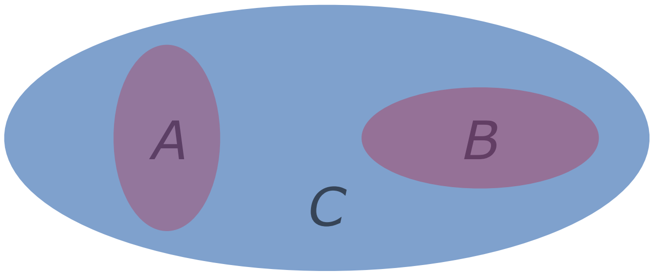


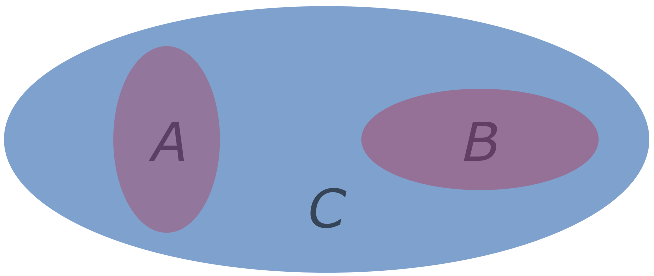
A+B in a **mixed state** $\rho_{AB} = \sum_i p_i \, |\psi_i(A,B)\rangle\langle\psi_i(A,B)|$

RECIPE
Entanglement of formation

- form all possible ensembles $\Omega = \{\, p_i, |\psi_i(A,B)\rangle\}$ that realize ρ_{AB}
- for each state in a given Ω , calculate $\mathcal{E}_i = -\mathrm{Tr}(\rho_A \log \rho_A)_i$
- find the minimal average entanglement over all ensembles

$$\mathcal{E}_F(\rho_{AB}) = \min_{\{\Omega\}} \sum_i p_i \, \mathcal{E}_i$$

Entanglement: some basics



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$$\mathcal{E}_F(\rho_{AB}) = \min_{\{\Omega\}} \sum_i p_i \, \mathcal{E}_i$$

Wootters, PRL 80, 2245 (1998) $\quad \mathcal{E}_F = h\left(\frac{1}{2}(1+\sqrt{1-C(\rho)^2})\right)$ for qubit systems

Why study entanglement of many-body quantum systems?

– Identify useful Hamiltonians to produce and control entangled states

– New schemes for quantum computing...

"topological quantum computing", "one-way quantum computing",...

– Get information about properties of complex ground state wave functions (without calculating them explicitly!)

– Identify and characterize ***quantum phase transitions (QPTs)***

A. Osterloh et al., Nature 416, 608 (2002)

T. Osborne and M. Nielsen, PRA 66, 032110 (2002)

What is a quantum phase transition?

Example: quantum Ising chain

$$H_I = -J \sum_{j=1}^{N-1} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z - gJ \sum_{j=1}^{N} \hat{\sigma}_j^x,$$

What is a quantum phase transition?

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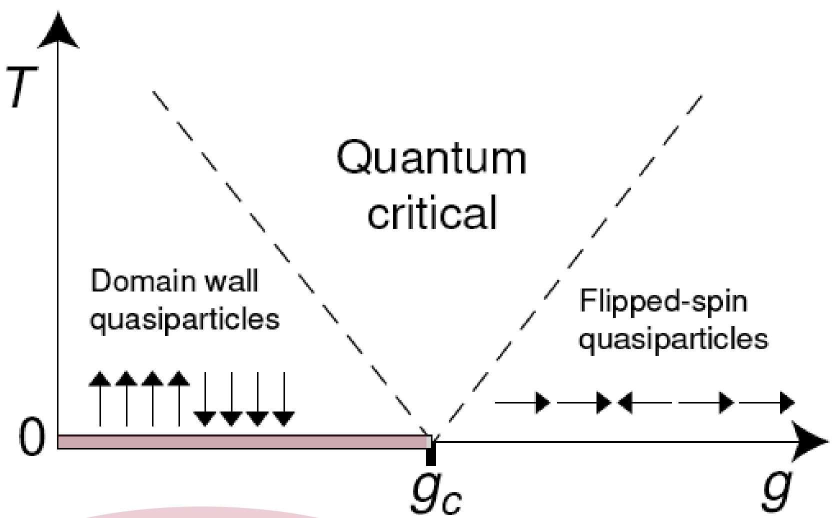

$|\uparrow\uparrow\uparrow\ldots\rangle$ or $|\downarrow\downarrow\downarrow\ldots\rangle$

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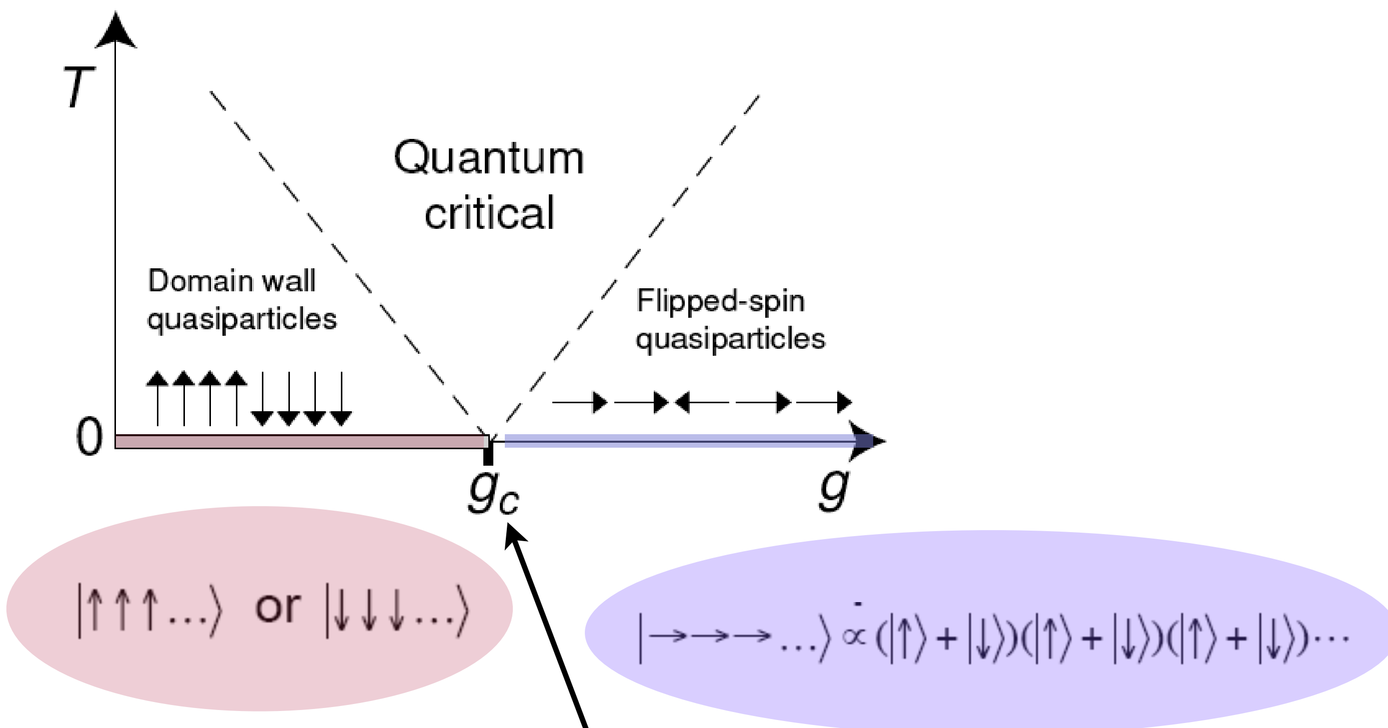
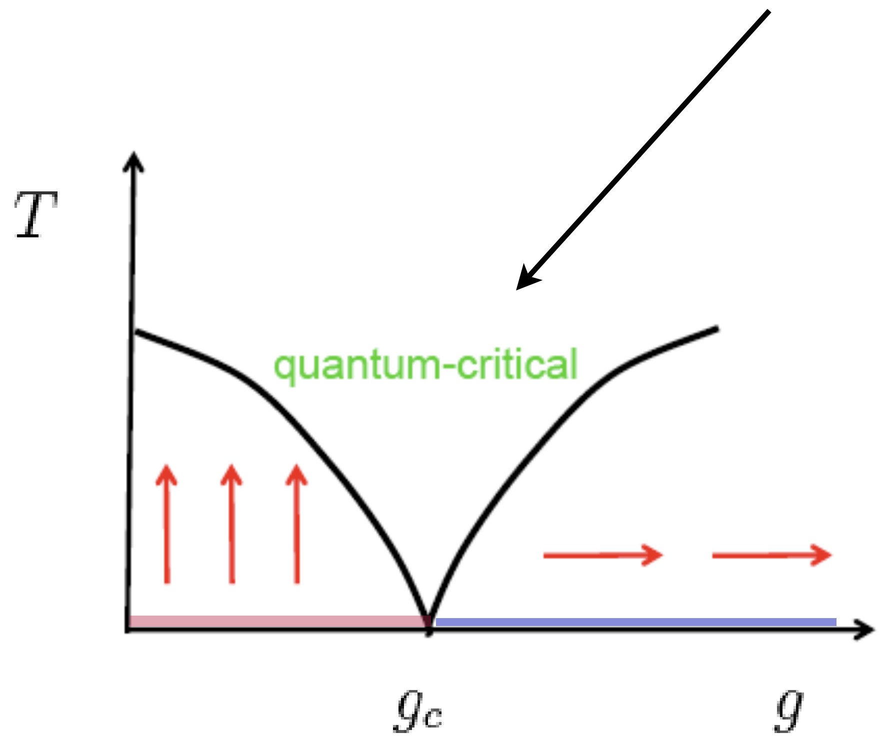


change of ground state at $g = g_c = 1$

What is a quantum phase transition?

breakdown of "text book" condensed matter physics:
anomalous **non-Fermi liquid behavior** ("heavy electrons", high T_c?,...)



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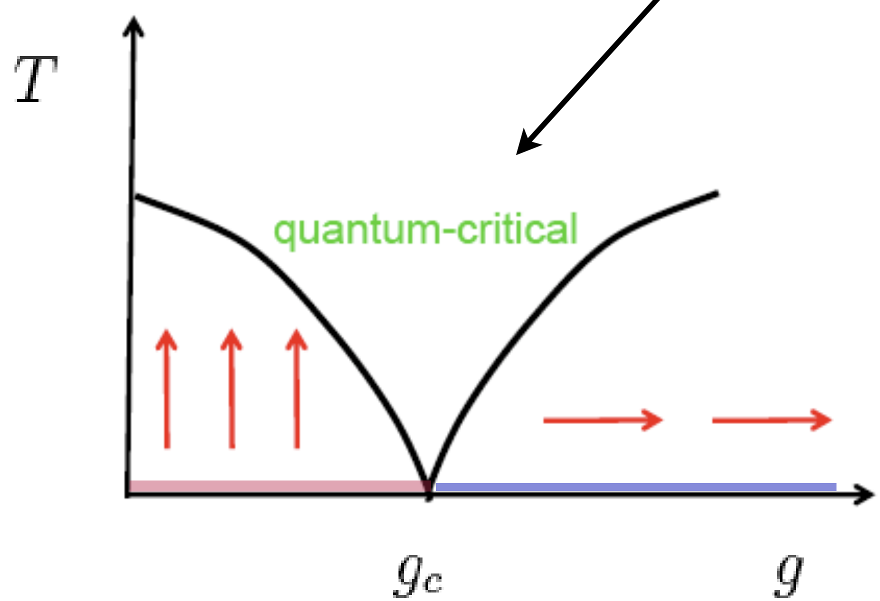
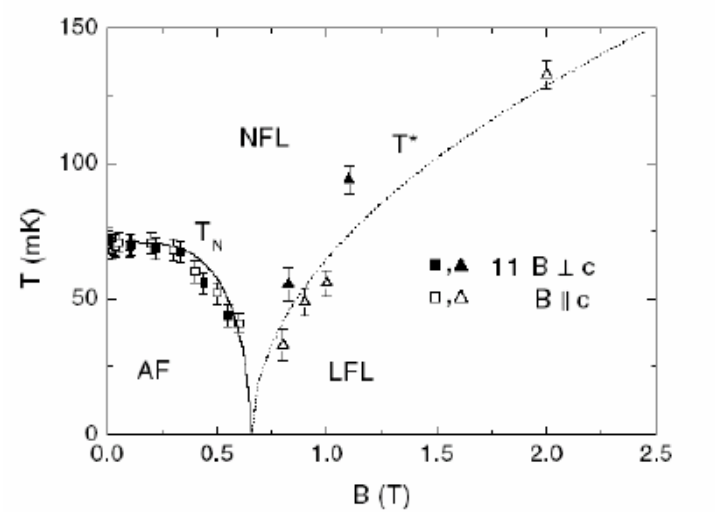


Quantum critical point in $YbRh_2Si_2$



Gegenwart et al., PRL 89:56402(2002)

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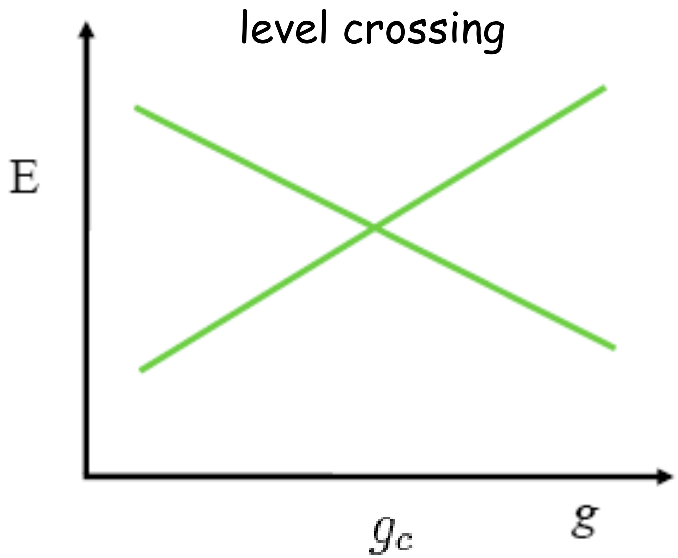
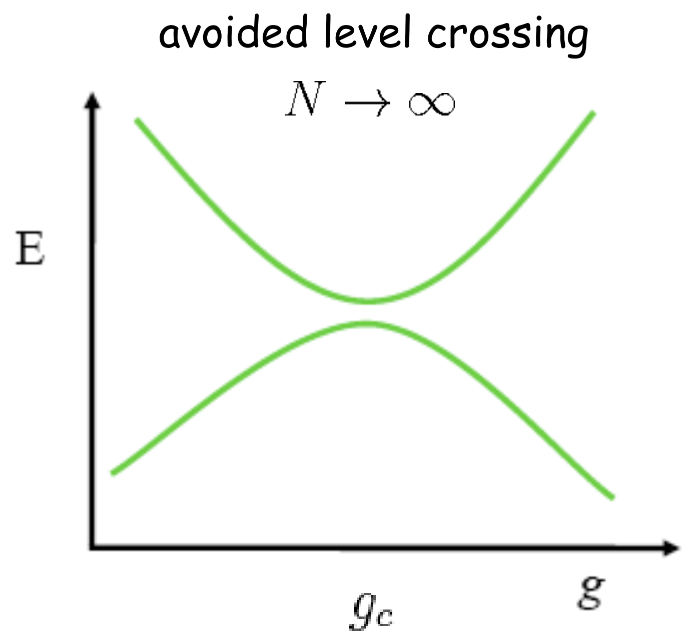


1st order QPT



2nd order QPT

non-analytic ground state energy → non-analytic density matrix

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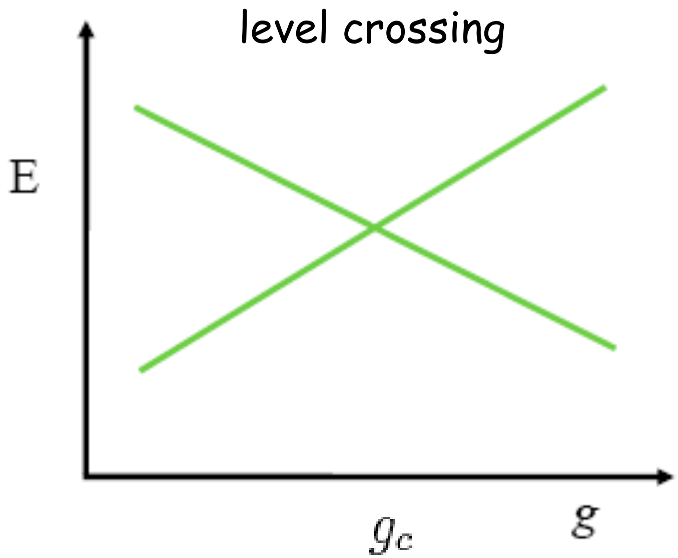

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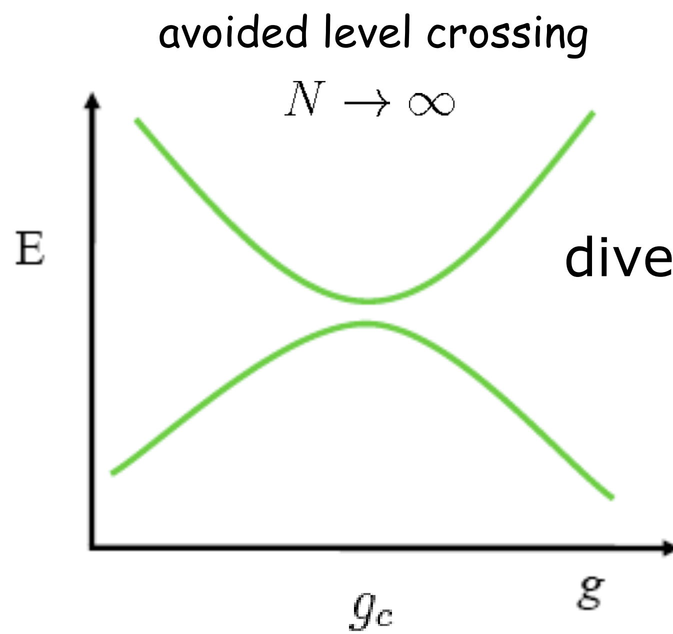

2nd order QPT

diverging correlation length

$\xi \sim |g - g_c|^{-\nu}$

scale invariance

non-analytic ground state energy → non-analytic density matrix

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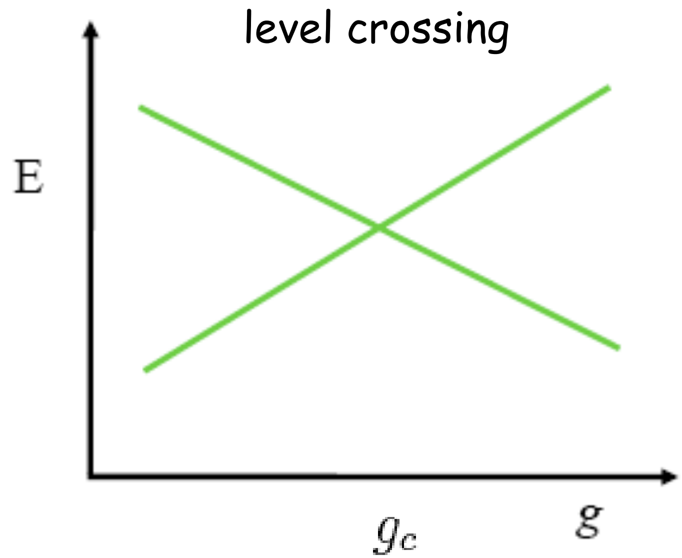

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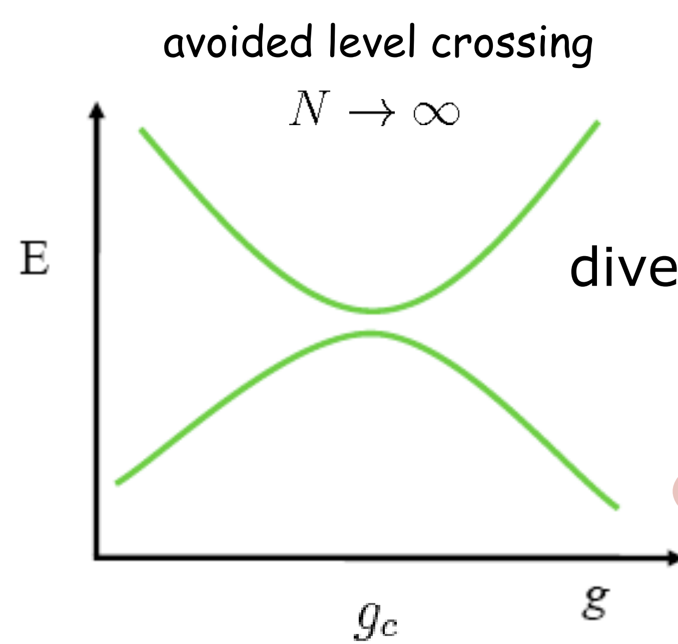

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How does this show up in the ground state entanglement?
What can we learn from it?

Spin-1/2 models (interacting qubits on a lattice)

Large body of (mostly numerical) *results* on spin chains and spin ladders (with or without frustration):

"A discontinuity [divergence] in the [first derivative of the] ground state concurrence between neighboring spins is associated with a first [second] order QPT."

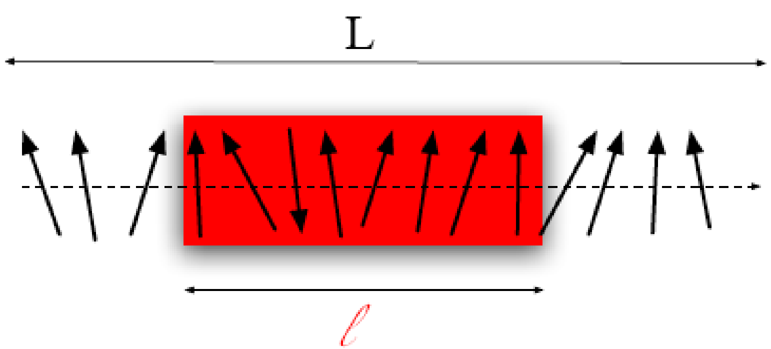
L.-A. Wu et al., PRL 93, 250404 (2004)

Expected from the theory of critical phenomena!
L. Campos Venuti et al., PRA 73, R010303 (2006)

Spin-1/2 models

Large body of results...

Scale invariance at criticality is reflected in the block (von Neumann) entropy



Example: $\Delta = 1$ transition in the XXZ chain

$$\mathcal{H} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

$$S_\ell = \frac{c}{3} \log_2 \left[\frac{L}{\pi a} \sin\left(\frac{\pi}{L}\ell\right)\right] + A$$

J.I. Latorre et al., Quant. Inf. and Comp. 4, 48 (2004)

General CFT setting:
P. Calabrese and J. Cardy, J. Stat. Mech., P06002 (2004)
V.E. Korepin, PRL 92, 096402 (2004)

Spin-1/2 models

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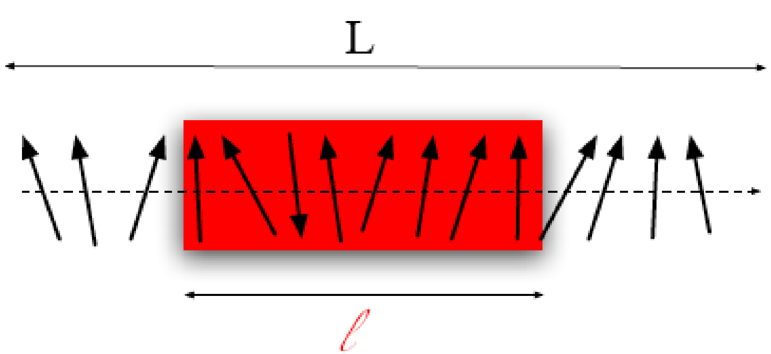

c-number labels the “universality class”
to which the critical theory belongs

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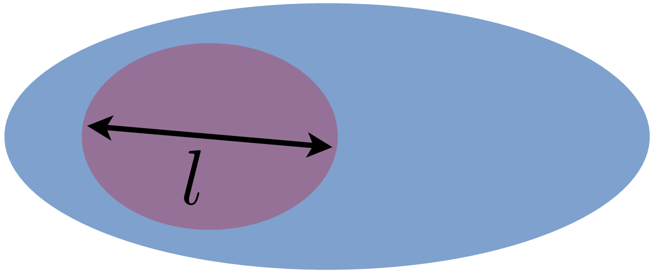
Spin-1/2 models

$$S_\ell = \frac{c}{3}\log_2\left[\frac{L}{\pi a}\sin\left(\frac{\pi}{L}\ell\right)\right] + A$$

The logarithmic scaling of the block entanglement in 1D critical spin systems violates the expected "area law" for entanglement

$$S_l \sim (l/a)^{d-1}$$



L. Bombelli et al. (1986)
M. Srednicki (1993)

→ "strong" entanglement in 1D critical systems!

Spin-1/2 models

Many other results for entanglement in spin-1/2 systems in 1D (***and*** 2D!)

- *effect of boundaries*
- *"topological states" on spin lattices*
  (Kitaev model, quantum dimer model,...)
- *impurities*
- *"quantum quenches"*
- *disorder*
- *.......*

For a review, see the special issue of J. Phys. A, soon to appear

Entanglement of itinerant particles?

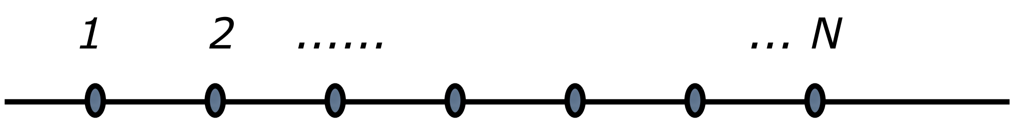
What about interacting **fermions** on a (1D) lattice?

Anti-symmetrization of fermion states:
physical Hilbert space lacks a direct product structure

How to define *entanglement?*

Use an occupation number representation!

P. Zanardi, PRA 65, 042101 (2002)



$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

$|n\rangle_j = |0\rangle_j, |\uparrow\rangle_j, |\downarrow\rangle_j \ |\uparrow\downarrow\rangle_j$ $\quad$ *j=1,...,N*

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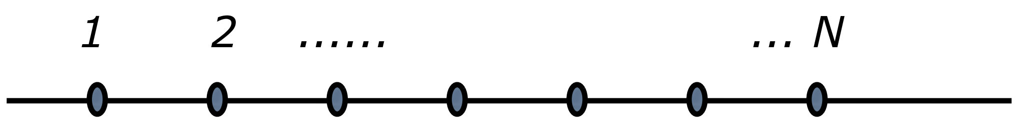

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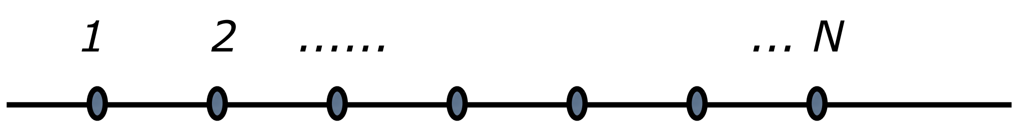

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

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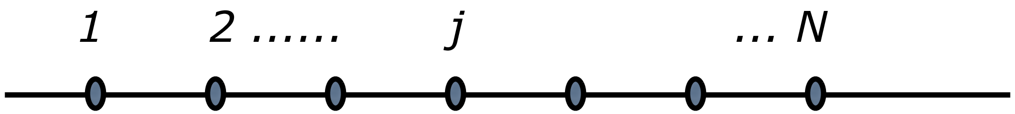
states describing the fermionic occupancies of the sites on a lattice

Entanglement of ~~Fermions~~ at Quantum Criticality

Fermionic entanglement

Introduce a translationally invariant Hamiltonian $\mathcal{H}(g) = \mathcal{H}_0 + g\Lambda$ that *conserves total spin and particle number*



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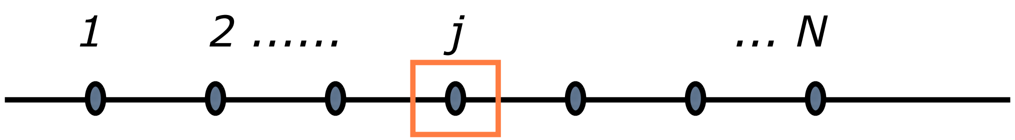

diagonal reduced density matrix

$$\rho_j = \sum_{\alpha=0,\uparrow,\downarrow} w_\alpha |\alpha\rangle_j\langle\alpha|_j + w_2|\uparrow\downarrow\rangle_j\langle\uparrow\downarrow|_j$$

expectation value of double occupancy

$$w_2 = \langle\psi_0|\hat{n}_{j\uparrow}\hat{n}_{j\downarrow}|\psi_0\rangle,$$

average single-site occupation

$$w_\uparrow = \langle\psi_0|\hat{n}_{j\uparrow}|\psi_0\rangle - w_2 = \frac{n}{2} + m - w_2,$$

average magnetization

$$w_\downarrow = \langle\psi_0|\hat{n}_{j\downarrow}|\psi_0\rangle - w_2 = \frac{n}{2} - m - w_2,$$

$$w_0 = 1 - n + w_2,$$

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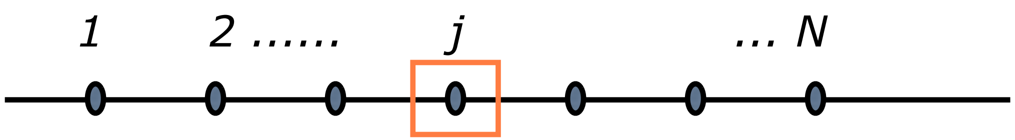

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single-site entanglement

$$\mathcal{E} = -w_0\log_2 w_0 - w_\uparrow\log_2 w_\uparrow - w_\downarrow\log_2 w_\downarrow - w_2\log_2 w_2$$

m, n, w_2

Fermionic entanglement

$$\mathcal{H}(g) = \mathcal{H}_0 + g\Lambda$$

Suppose that

$\partial^{k-1}\mathcal{E}/\partial g^{k-1}$ is singular at $g = g_c$

$\Downarrow$

g = magnetic field
chemical potential
local interaction

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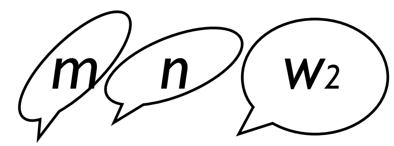
$\partial^{k-1}\mathcal{O}_g/\partial g^{k-1}$ singular at $g=g_c$



$$\mathcal{O}_g \equiv [\langle \psi_0 | \Lambda | \psi_0 \rangle \text{—regular terms}]$$

Fermionic entanglement

$$\mathcal{H}(g) = \mathcal{H}_0 + g\Lambda$$

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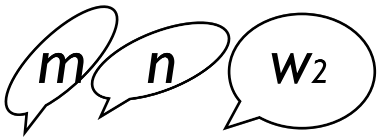
local interaction

$\partial^{k-1}\mathcal{O}_g/\partial g^{k-1}$ singular at $g = g_c$



$$\mathcal{O}_g \equiv [\langle \psi_0 | \Lambda | \psi_0 \rangle \text{—regular terms}]$$

Hellman-Feynman theorem

$\partial^k e_0/\partial g^k$ singular at $g = g_c$

$$e_0 = \langle \psi_0 | \mathcal{H}(g) | \psi_0 \rangle$$

Fermionic entanglement

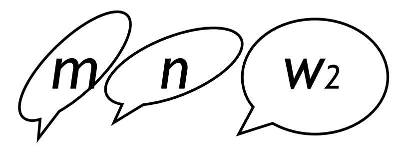
$$\mathcal{H}(g) = \mathcal{H}_0 + g\Lambda$$

Suppose that

$\partial^{k-1}\mathcal{E}/\partial g^{k-1}$ is singular at $g = g_c$

g = magnetic field
chemical potential
local interaction

$\partial^{k-1}\mathcal{O}_g/\partial g^{k-1}$ singular at $g = g_c$



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Hellman-Feynman theorem

$\partial^k e_0/\partial g^k$ singular at $g = g_c$

$$e_0 = \langle\psi_0|\mathcal{H}(g)|\psi_0\rangle$$

k:th order QPT at $g = g_c$

Fermionic entanglement

Divergence/discontinuity in the *k-1*:st derivative of the single-site entanglement → *k*:th order QPT

D. Larsson and H. Johannesson, PRA 73, 042320 (2006)

Fermionic entanglement

Divergence/discontinuity in the *k-1*:st derivative of the single-site entanglement

?

k:th order QPT

D. Larsson and H. Johannesson, PRA 73, 042320 (2006)

doesn't apply to

– Kosterlitz-Thouless type transitions (essential singularities!)

– QPTs out of topologically ordered phases (no local order parameters!)

X.-G. Wen, PRB 65, 165113 (2002)

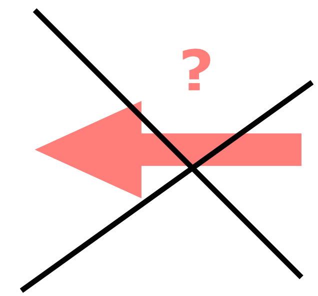
– QPTs with equal weight of the available local states

D. Larsson and H. Johannesson, PRA 73, 042320 (2006)

kills off the non-analyticity in the single-site entanglement!

Fermionic entanglement

Divergence/discontinuity in the *k-1*:st derivative of the single-site entanglement



k:th order QPT

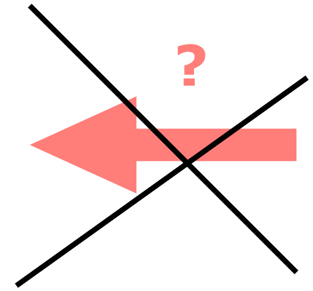
D. Larsson and H. Johannesson, PRA 73, 042320 (2006)

equal weight of the available local states kills off the non-analyticity

$$
\begin{aligned}
\frac{\partial^{k-1}\mathcal{E}}{\partial g^{k-1}} = &-\left(\frac{\partial^{k-1}}{\partial g^{k-1}}\left[\frac{n}{2}+m-w_2\right]\right)\log_2\left(\frac{n}{2}+m-w_2\right)\\
&-\left(\frac{\partial^{k-1}}{\partial g^{k-1}}\left[\frac{n}{2}-m-w_2\right]\right)\log_2\left(\frac{n}{2}-m-w_2\right)\\
&+\left(\frac{\partial^{k-1}}{\partial g^{k-1}}[n-w_2]\right)\log_2(1-n+w_2)\\
&-\frac{\partial^{k-1}w_2}{\partial g^{k-1}}\log_2(w_2)\\
&+\text{terms containing lower-order derivatives}
\end{aligned}
$$

Fermionic entanglement

Divergence/discontinuity in the *k-1*:st derivative of the single-site entanglement



k:th order QPT

equal weight of the available local states kills off the non-analyticity

$$\partial\mathcal{E}/\partial g = 0$$

extremum of the single-site entanglement

$$
\begin{aligned}
\frac{\partial^{k-1}\mathcal{E}}{\partial g^{k-1}} = &-\left(\frac{\partial^{k-1}}{\partial g^{k-1}}\left[\frac{n}{2}+m-w_2\right]\right)\log_2\!\left(\frac{n}{2}+m-w_2\right)\\
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A case study: the 1D Hubbard model

$$\mathcal{H} = -t\sum_{\substack{j=1\\ \delta=\pm 1}}^{L} c^{\dagger}_{j\alpha}c_{j+\delta\alpha} + U\sum_{j=1}^{L} n_{j\uparrow}n_{j\downarrow} - \mu_B H\sum_{j=1}^{L} S^z_j - \mu\sum_{j=1}^{L}(n_{j\uparrow} + n_{j\downarrow})$$

– minimal model for correlated fermions

– exactly solvable by *Bethe Ansatz*

Lieb and Wu, PRL 20, 1445 (1968)

– exhibits QPTs controlled by *U, H,* and μ

– realized in optical lattices of ultracold fermionic gases

Moritz et al., PRL 94, 210401 (2005)

Jördens et al., Nature 455, 204 (2008)

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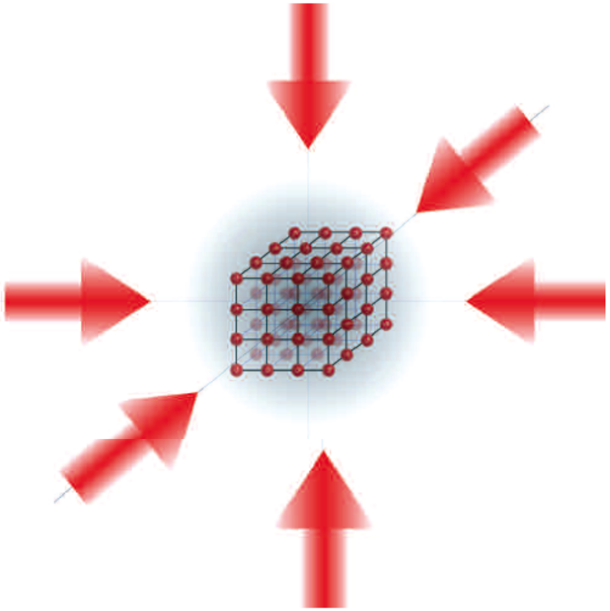
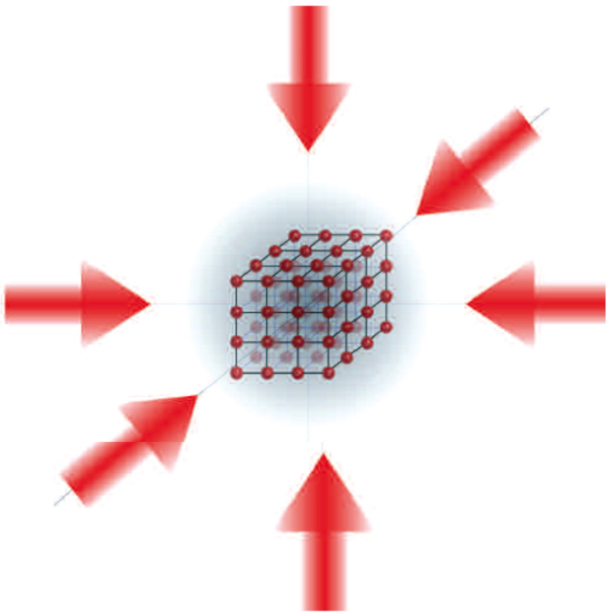
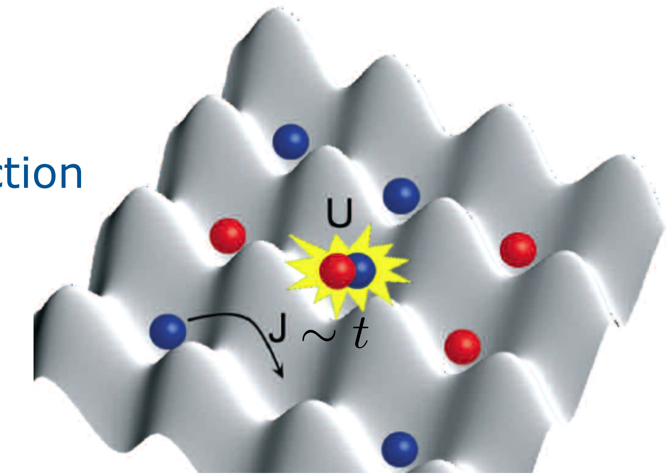


ultracold gas of fermionic atoms (^{40}K)
trapped in an optical lattice produced
by pairs of opposite laser beams

A case study: the 1D Hubbard model



2D intersection



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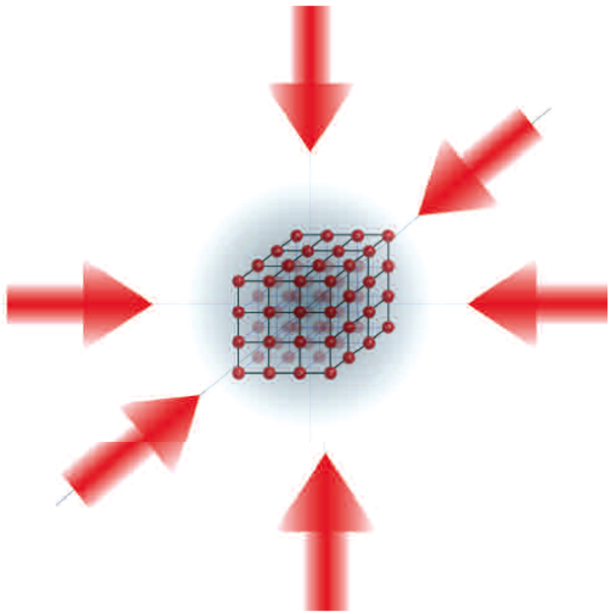
ultracold gas of fermionic atoms (^{40}K) trapped in an optical lattice produced by pairs of opposite laser beams

tunneling t and interaction U (determined by a Feshbach resonance) can be tuned!

A case study: the 1D Hubbard model



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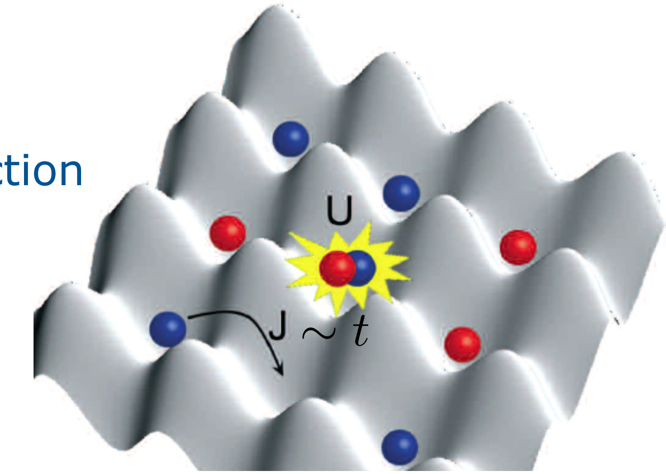

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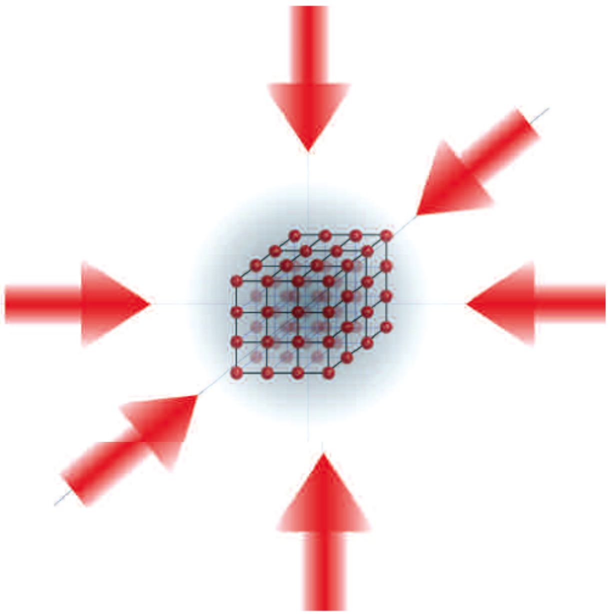
1D geometry created when two pairs of the laser beams have very high intensity (suppresses tunneling along these beam directions)

1D Hubbard model

For a review, see
M. Köhl and T. Esslinger, Europhysics News 37/2, 18 (2006)

A case study: the 1D Hubbard model



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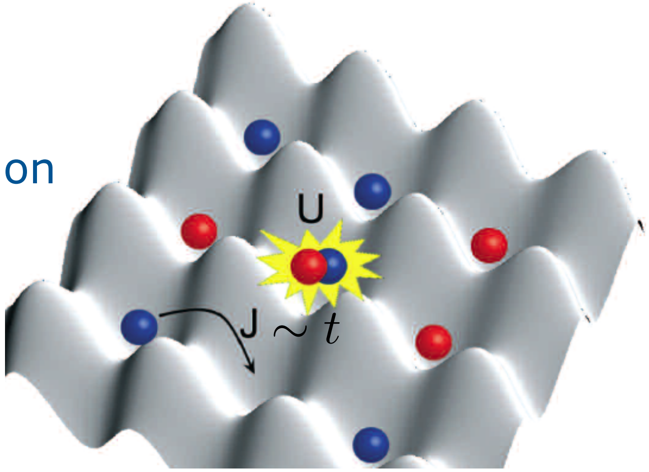

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1D Hubbard model (with a confining potential)

For a review, see
M. Köhl and T. Esslinger, Europhysics News 37/2, 18 (2006)

A case study: the 1D Hubbard model

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QPTs at $U = U_c$, $H = H_c$ and $\mu = \mu_c$

How to extract the single-site entanglement?

Recipe:

write $\mathcal{E} = -w_0 \log_2 w_0 - w_\uparrow \log_2 w_\uparrow - w_\downarrow \log_2 w_\downarrow - w_2 \log_2 w_2$

calculate w_0, $w_\uparrow$, $w_\downarrow$, w_2 from the groundstate energy using the Hellman-Feynman theorem

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from the Bethe Ansatz solution of the Hubbard model

E.H. Lieb and F.Y. Wu, PRL 20, 1445 (1968)

Mott-Hubbard transition at half-filling *(n=1)*

U>0, H = 0, control parameter: μ

$$\mathcal{H} = -t \sum_{\substack{j=1 \\ \delta=\pm 1}}^{L} c^{\dagger}_{j\alpha} c_{j+\delta\alpha} + U \sum_{j=1}^{L} n_{j\uparrow} n_{j\downarrow} - \mu_B H \sum_{j=1}^{L} S^z_j - \mu \sum_{j=1}^{L} (n_{j\uparrow} + n_{j\downarrow})$$

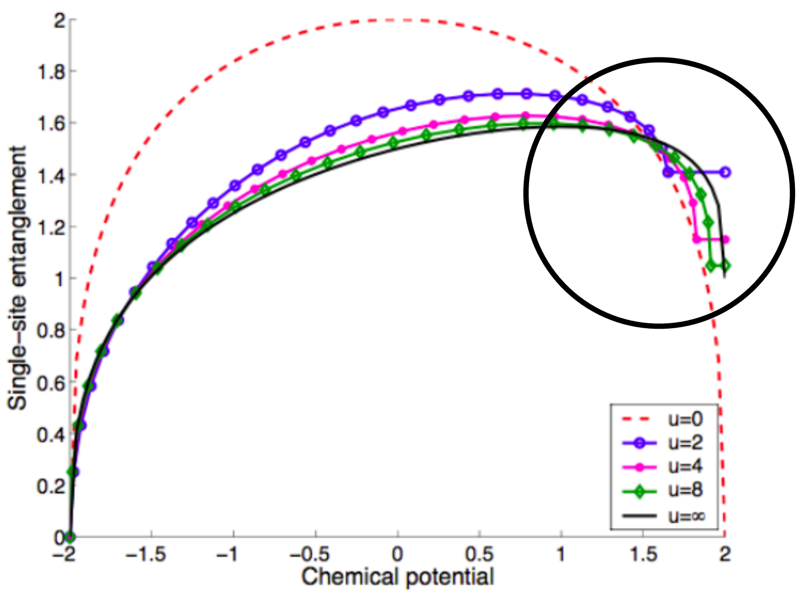
$u \longrightarrow \infty$ limit

$$\partial \mathcal{E} / \partial \mu = \chi_c (\ln |\mu - \mu_c| + \text{const.}) / (2 \ln 2)$$

$$\chi_c = \partial n / \partial \mu \sim |\mu - \mu_c|^{-1/2}$$

finite u

$$\partial \mathcal{E} / \partial \mu = \chi_c \, C(u)$$



D. Larsson and H. Johannesson
PRL 95, 196406 (2005);
ibid. 96, 169906(E) (2006)

A case study: the 1D Hubbard model

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1/u-expansion of the ground state energy
J. Carmelo and D. Baeriswyl, Phys. Rev. B 37, 7541 (1988)

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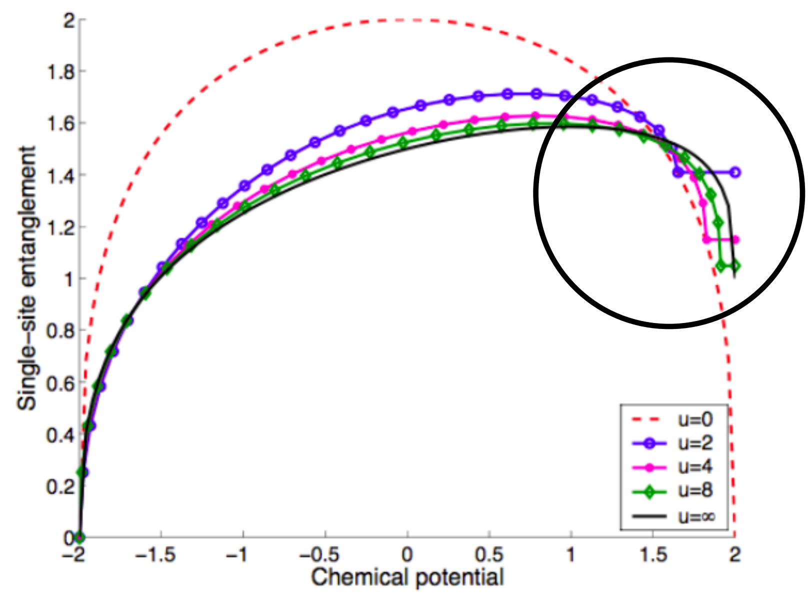

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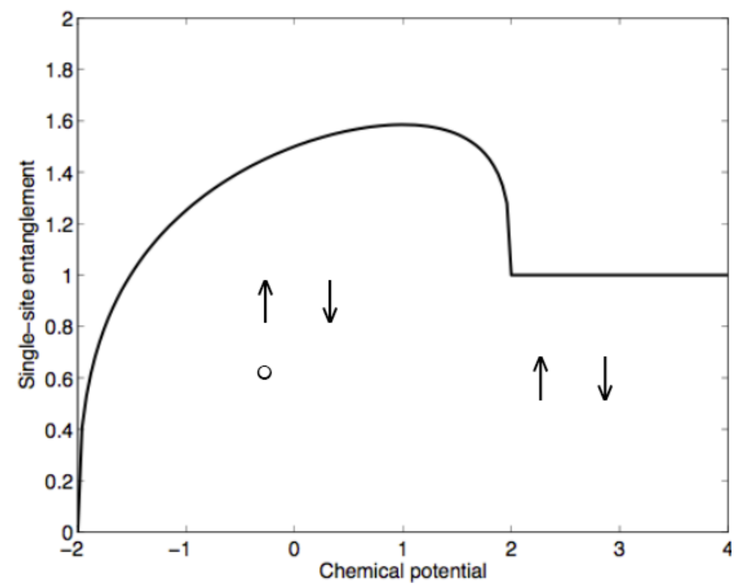
$u \longrightarrow \infty$ limit

$$\partial\mathcal{E}/\partial\mu = \chi_c(\ln|\mu-\mu_c| + \text{const.})/(2\ln 2)$$

logarithmic correction

change of “effective” local dimension



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A case study: the 1D Hubbard model

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finite u

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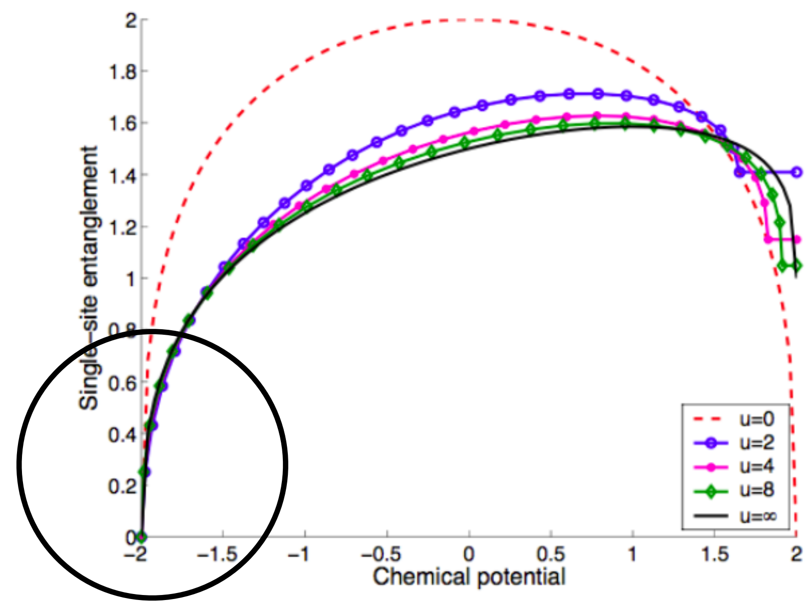


empty lattice transition

same scaling as Mott-Hubbard in
$u \longrightarrow \infty$ limit

D. Larsson and H. Johannesson
PRL 95, 196406 (2005);
ibid. 96, 169906(E) (2006)

A case study: the 1D Hubbard model

repulsive interaction

Magnetic transitions at half-filling (n=1, μ fixed)

U>0, control parameter: *H*

$$\mathcal{H} = -t \sum_{\substack{j=1 \\ \delta=\pm 1}}^{L} c_{j\alpha}^{\dagger} c_{j+\delta\alpha} + U \sum_{j=1}^{L} n_{j\uparrow} n_{j\downarrow} - \mu_B H \sum_{j=1}^{L} S_j^z - \mu \sum_{j=1}^{L} (n_{j\uparrow} + n_{j\downarrow})$$

saturation at $H=h_{c2}$

any $u>0$

$$\frac{\partial \mathcal{E}}{\partial h} = \frac{C}{2\ln(2)} \chi_S (\ln|h - h_{c2}| + \text{const})$$

$$h \longrightarrow h_{c2-} \quad h_{c2} = 4(\sqrt{1+u^2} - u)$$

$$2\pi\chi_S = (4 + 4u^2)^{1/4} |h - h_{c2}|^{-1/2}$$

$$C = 2 - u/\sqrt{1+u^2},$$

D. Larsson and H. Johannesson
PRL 95, 196406 (2005)

A case study: the 1D Hubbard model

attractive interaction

Magnetic transitions at half-filling (n=1, μ fixed)

U<0, control parameter: *H*

$$\mathcal{H} = -t \sum_{\substack{j=1 \\ \delta=\pm 1}}^{L} c_{j\alpha}^{\dagger} c_{j+\delta\alpha} + U \sum_{j=1}^{L} n_{j\uparrow} n_{j\downarrow} - \mu_B H \sum_{j=1}^{L} S_j^z - \mu \sum_{j=1}^{L} (n_{j\uparrow} + n_{j\downarrow})$$

onset of magnetization at $H=h_{c1}$

saturation at $H=h_{c2}$

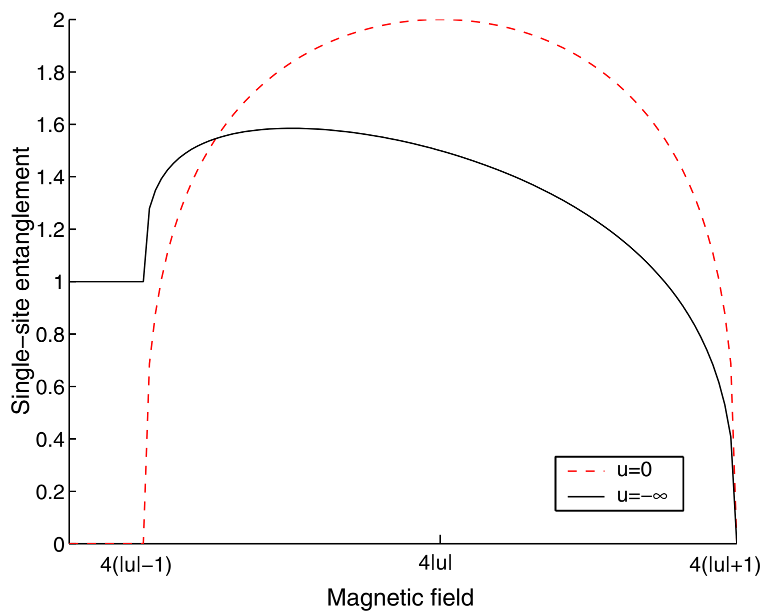
$|u| \gg 1$

$$\frac{\partial \mathcal{E}}{\partial h} = (-1)^i \frac{\chi_{Si}}{\ln(2)} (\ln|h - h_{ci}| + \text{const})$$

$i = 1, 2$

$h \longrightarrow h_{c1+} \qquad h \longrightarrow h_{c2-}$

$$\chi_{Si} = (32\pi^2 |h - h_{ci}|)^{-1/2}$$



D. Larsson and H. Johannesson
PRL 95, 196406 (2005)

A case study: the 1D Hubbard model

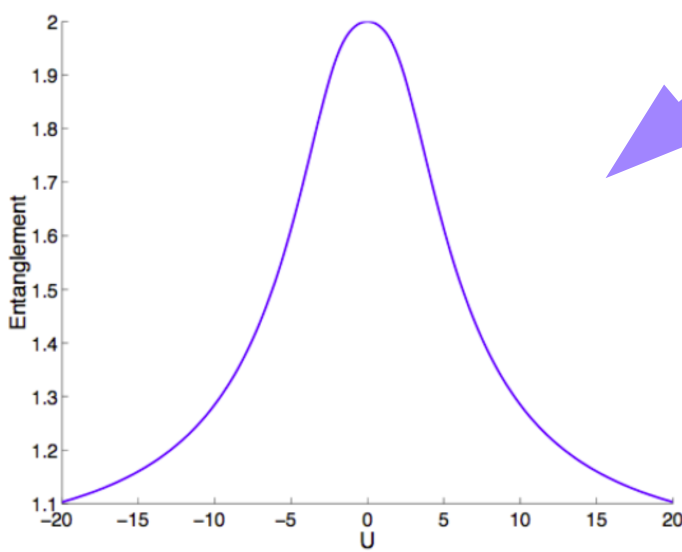
Mott-Hubbard transition at *U=0*

control parameter: U ($H=0$, half-filling)

$$\mathcal{H} = -t \sum_{\substack{j=1 \\ \delta=\pm 1}}^{L} c_{j\alpha}^{\dagger} c_{j+\delta\alpha} + U \sum_{j=1}^{L} n_{j\uparrow} n_{j\downarrow} - \mu_B H \sum_{j=1}^{L} S_j^z - \mu \sum_{j=1}^{L} (n_{j\uparrow} + n_{j\downarrow})$$

– QPT of *infinite order* *(Kosterlitz-Thouless type)*
Metzner and Vollhardt, PRB 39, 4462 (1989)

– $|n\rangle_j = |0\rangle_j, |\uparrow\rangle_j, |\downarrow\rangle_j$ $|\uparrow\downarrow\rangle_j$ $j=1,\ldots,N$ equally weighted at $U=0$



maximum at the QPT!

S.-J. Gu et al., PRL 93, 086402 (2004)

Another case study

Hubbard model with long-range hopping

F. Gebhard and A.E. Ruckenstein, PRL 68, 244 (1992)

$$H = \sum_{\substack{\ell \neq m=1 \\ \sigma=\uparrow,\downarrow}}^{L} t_{\ell m} \hat{c}^{\dagger}_{\ell\sigma} \hat{c}_{m\sigma} + u \sum_{l=1}^{L} \hat{n}_{\ell\uparrow} \hat{n}_{\ell\downarrow}$$

$$t_{\ell m} = i(-1)^{(l-m)}(l-m)^{-1}$$

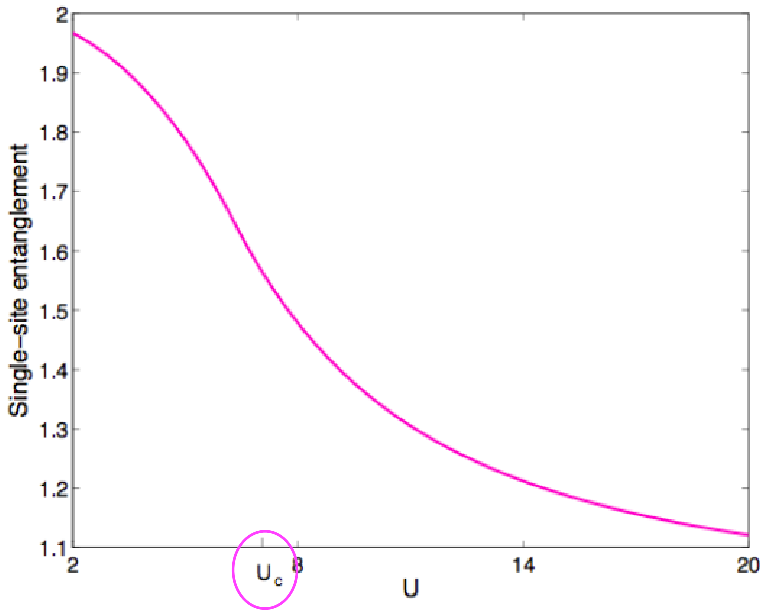
Mott-Hubbard transition

control parameter: u (half-filling)

$\partial^2 \mathcal{E} / \partial u^2$ discontinuous at $u_c = 2\pi$

↓

third-order QPT



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$$t_{\ell m} = i(-1)^{(l-m)}(l-m)^{-1}$$

Mott-Hubbard transition

control parameter: μ $(u > u_c)$

$\partial\mathcal{E}/\partial\mu$ discontinuous at $\mu_c = \pi$

second-order QPT

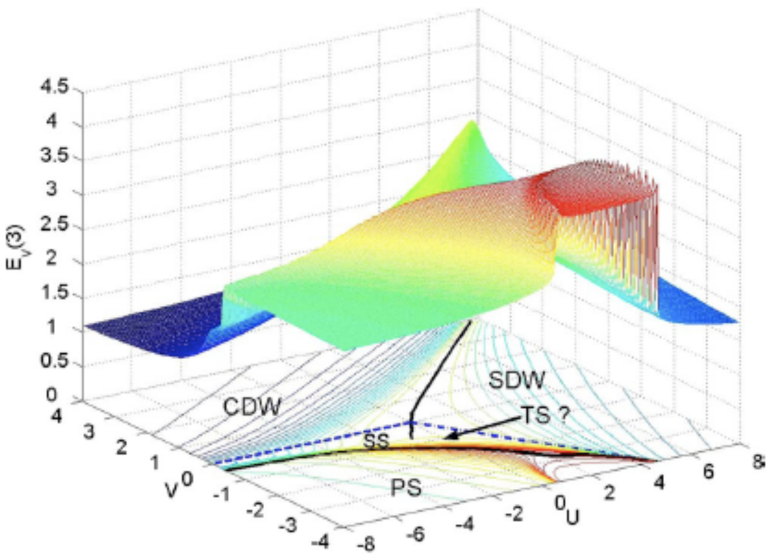
with logarithmic correction for $\mu \to \mu_{c-}$

(suppression of empty local states)

D. Larsson and H. Johannesson, PRA 73, 042320 (2006)

Yet another case study: the **extended** 1D Hubbard model

$$\mathcal{H} = -\sum_{\sigma,i,\delta} c^{\dagger}_{i,\sigma} c_{i+\delta,\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} + V \sum_i n_i n_{i+1}$$



Phase diagram from numerical study of the block entanglement (n=1)

S.-J. Gu et al, Phys. Rev. Lett. 93, 086402 (2004)

S.-S. Deng et al. Phys. Rev. B 74, 045103 (2006)

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n=1/2, $V \to \infty$

K. Penc and F. Mila, Phys. Rev. B 49, 9670 (1994)

Analytic results for entanglement scaling:

$$\frac{\partial \mathcal{E}}{\partial U} \approx \frac{1}{4\sqrt{2}\pi} \frac{1}{\sqrt{|U_c - U|}} \{\log_2(|U_c - U|)\}$$

H. Johannesson and D. Larsson, Low Temp. Phys. 33, 1232 (2007)

U_c = – 4: *metal to superconductor transition*

U_c = 4: *metal–insulator transition*

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n=1/2, $U \to \infty$

Jordan-Wigner

Mott-Hubbard transition at V = 2 $\longleftrightarrow$ AFM Ising transition in the s=1/2 XXZ chain

Analytic two-site entanglement with a *maximum* at V = 2
due to the particular weighting of the local states!

Another case study: the **extended** 1D Hubbard model

$$\mathcal{H} = -\sum_{\sigma,i,\delta} c^\dagger_{i,\sigma} c_{i+\delta,\sigma} + U\sum_i n_{i\uparrow}n_{i\downarrow} + V\sum_i n_i n_{i+1}$$

n=1/2, $V \to \infty$

K. Penc and F. Mila, Phys. Rev. B 49, 9670 (1994)

Analytic results for entanglement scaling:

$$\frac{\partial \mathcal{E}}{\partial U} \approx \frac{1}{4\sqrt{2}\pi}\frac{1}{\sqrt{|U_c - U|}}\{\log_2(|U_c - U|)\}$$

H. Johannesson and D. Larsson, Low Temp. Phys. 33, 1232 (2007)

U_c = – 4: *metal to superconductor transition*

U_c = 4: *metal–insulator transition*

n=1/2, $U \to \infty$

D. Larsson and H. Johannesson, in progress

Jordan-Wigner

Mott-Hubbard transition at V = 2 ⟷ AFM Ising transition in the s=1/2 XXZ chain

Analytic two-site entanglement with a *maximum* at V = 2
due to the particular weighting of the local states!

Entanglement in inhomogeneous fermion systems

Entanglement in **inhomogeneous** fermion systems

boundaries, interfaces, impurities, defects,
spatial modulations of system parameters and external fields...

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If quantum information processing is ever to become reality we must be able to quantify entanglement in systems with inhomogeneities!

Entanglement in inhomogeneous fermion systems

Example: Hubbard chain with a local potential

$$\hat{H} = -t\sum_{i,\sigma}(c^{\dagger}_{i\sigma}c_{i+1,\sigma} + H.c.) + U\sum_{i}\hat{n}_{i\uparrow}\hat{n}_{i\downarrow} + \sum_{i\sigma} v_{i\sigma}\hat{n}_{i\sigma}$$

e.g. a ***confining potential*** *in an optical lattice of ultracold fermionic atoms*

The Hohenberg-Kohn theorem guarantees that the entanglement is a functional of the ground-state density n[x]

$$\mathcal{E}[n(x)]$$

Local-density approximation (LDA) for the entanglement:

V. V. Franca and K. Capelle, Phys. Rev. Lett. 100, 070403 (2008)

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$$v_{i\sigma}$$

from *Bethe–Ansatz* LDA

N. A. Lima et al., Phys. Rev. Lett. 90, 146402 (2003)

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A harmonic confining potential strongly reduces the entanglement

V. V. Franca and K. Capelle, Phys. Rev. Lett. 100, 070403 (2008)

How do *local* potentials (impurites) influence the entanglement?
Entanglement close to a *boundary*?
Entanglement scaling at criticality in the presence of *inhomogeneities?*
...and many other questions...

A rich and important field of study!

Summary

A generic finite-order QPT in a spin-1/2 fermionic lattice system driven by a change of a local interaction or an external field can be identified and characterized via the *single-site entanglement* *(with some caution!)*

example

second-order QPTs in the 1D Hubbard model

$$\frac{\partial \mathcal{E}}{\partial h} = \frac{C}{2\ln(2)}\chi_S(\ln|h - h_{c2}| + \mathrm{const})$$

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X logarithmic correction when the number of accessible local states change at the transition

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Implications for the theory of QPTs / quantum information?

More work needed!