DIAS Summer School
Quantum Integrable Systems and Quantum Information
Dungarvan, 18 - 24 August 2013

Quantum Phase Transitions in Integrable Systems: A Quantum Information Perspective

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What is a quantum phase transition?

What is a quantum phase transition?

Let’s first recall the notion of a classical phase transition...

Frozen waterfall, Eldfjord, Norway

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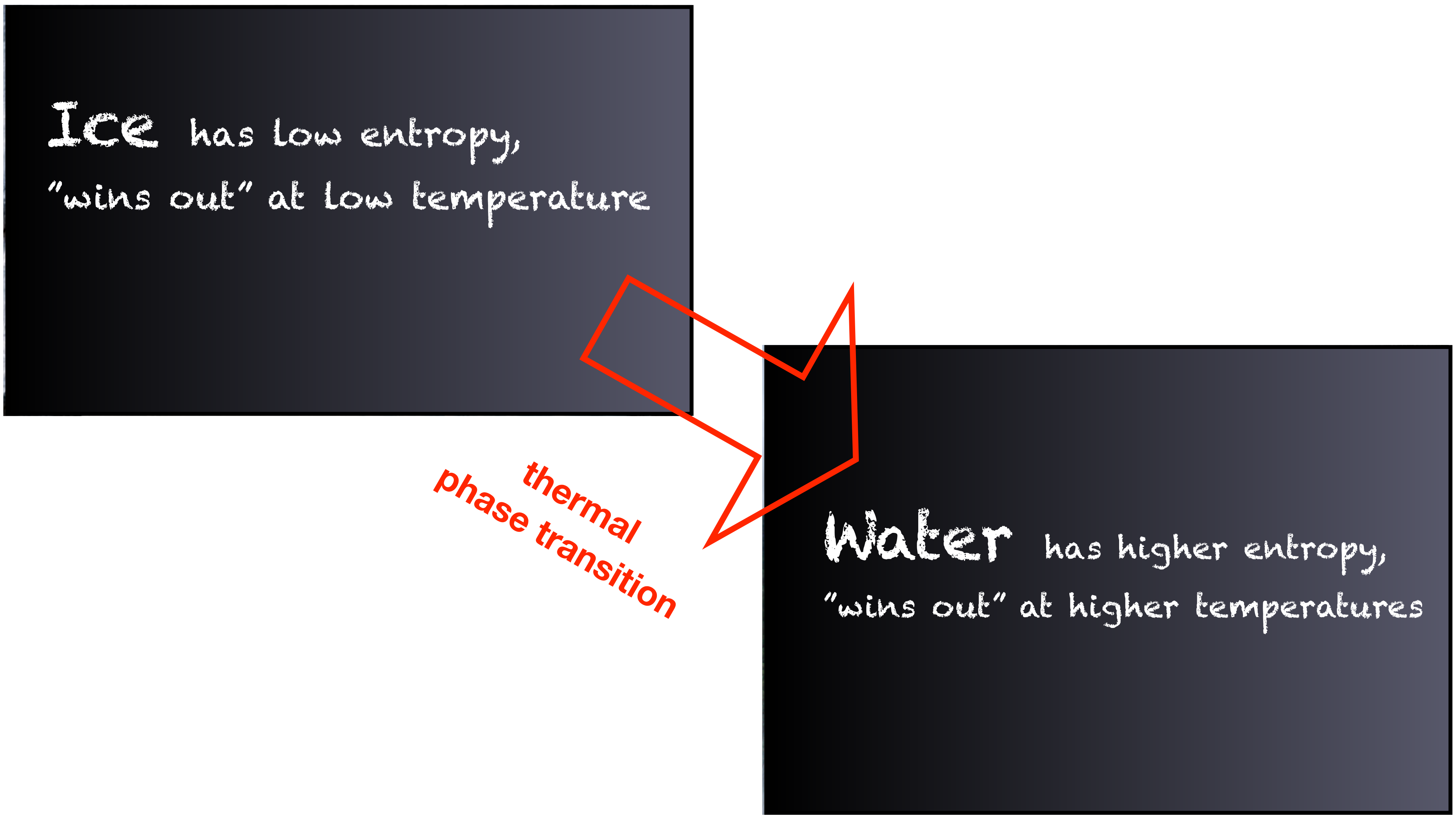
Frozen waterfall, Eldfjord, Norway

Nature tries to minimize the free energy $F = E - TS$



Nature tries to minimize the free energy $F = E - TS$

$$S = k_B \ln \Omega$$

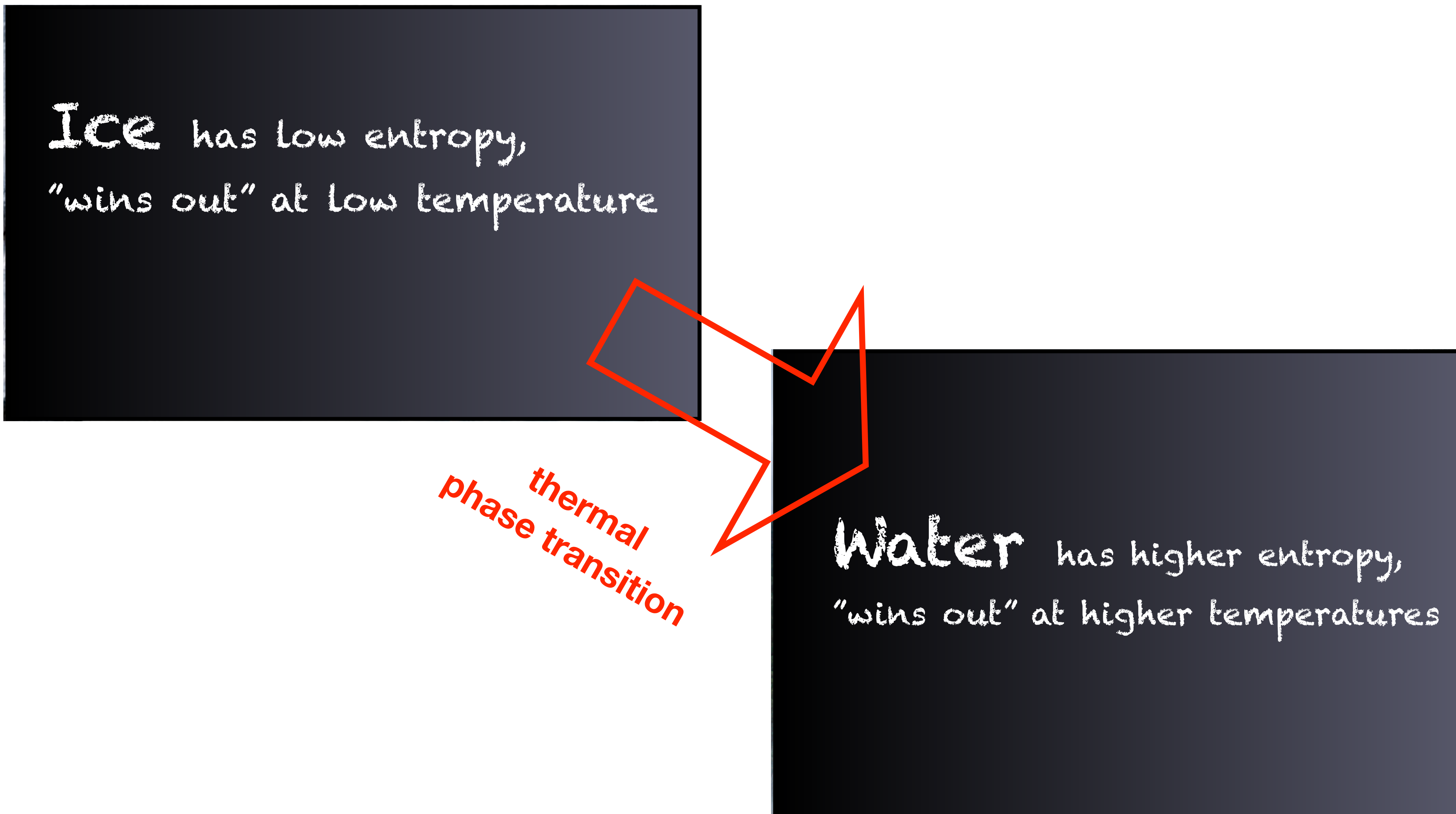
# of microscopic states compatible with a given macroscopic state

Ice has low entropy,
"wins out" at low temperature

thermal
phase transition

Water has higher entropy,
"wins out" at higher temperatures

Nature tries to minimize the free energy $F = E - TS$



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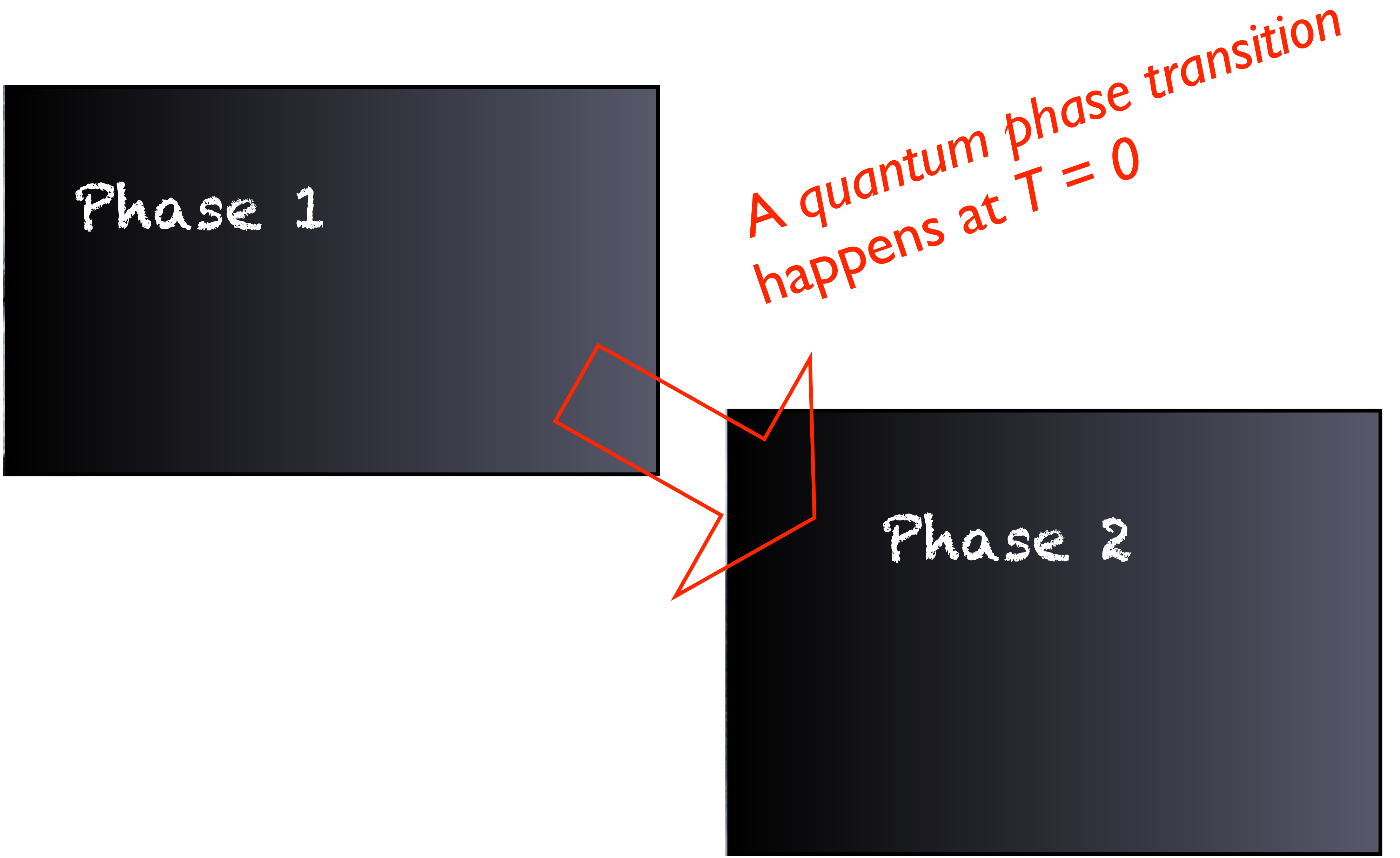
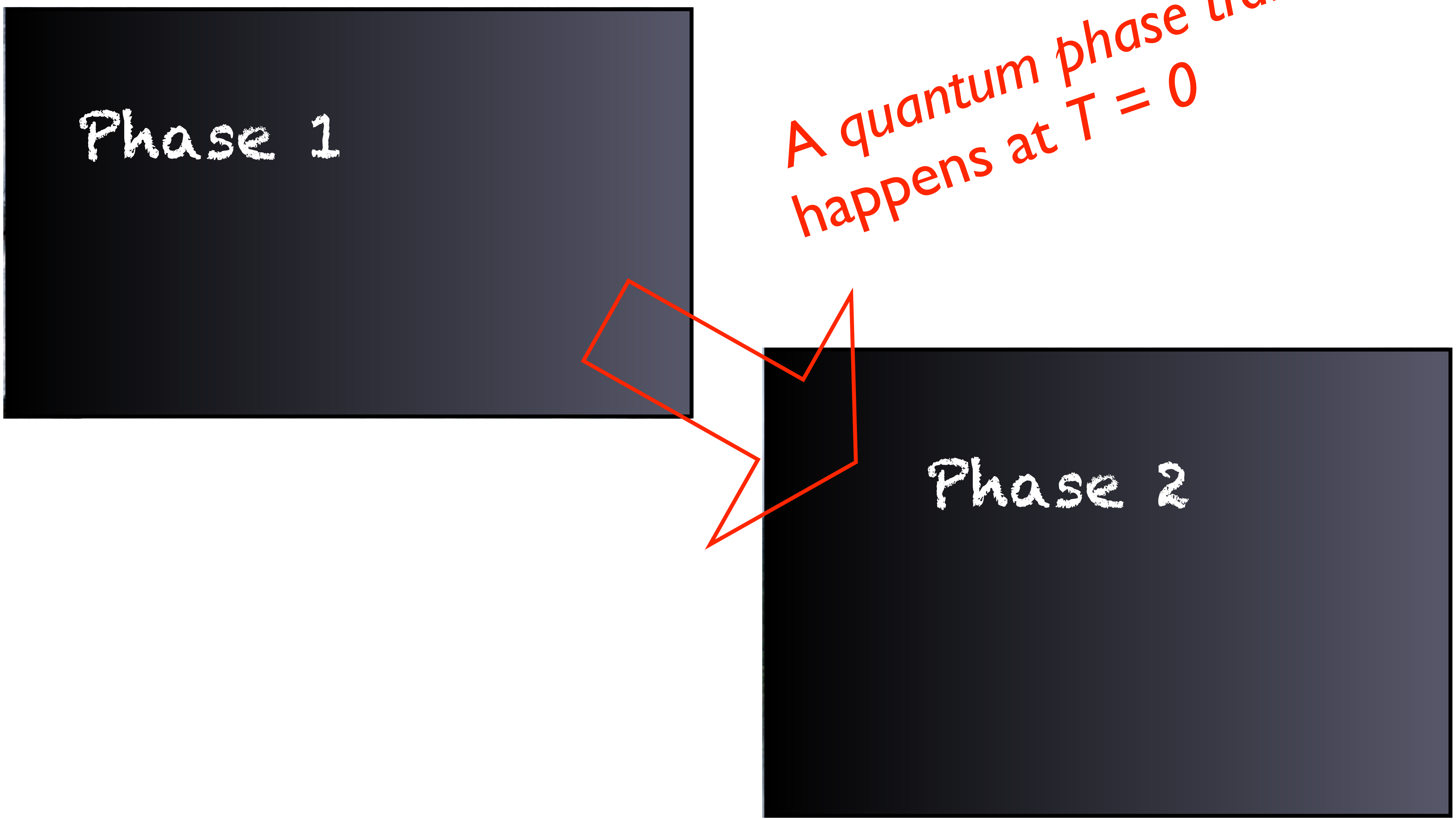
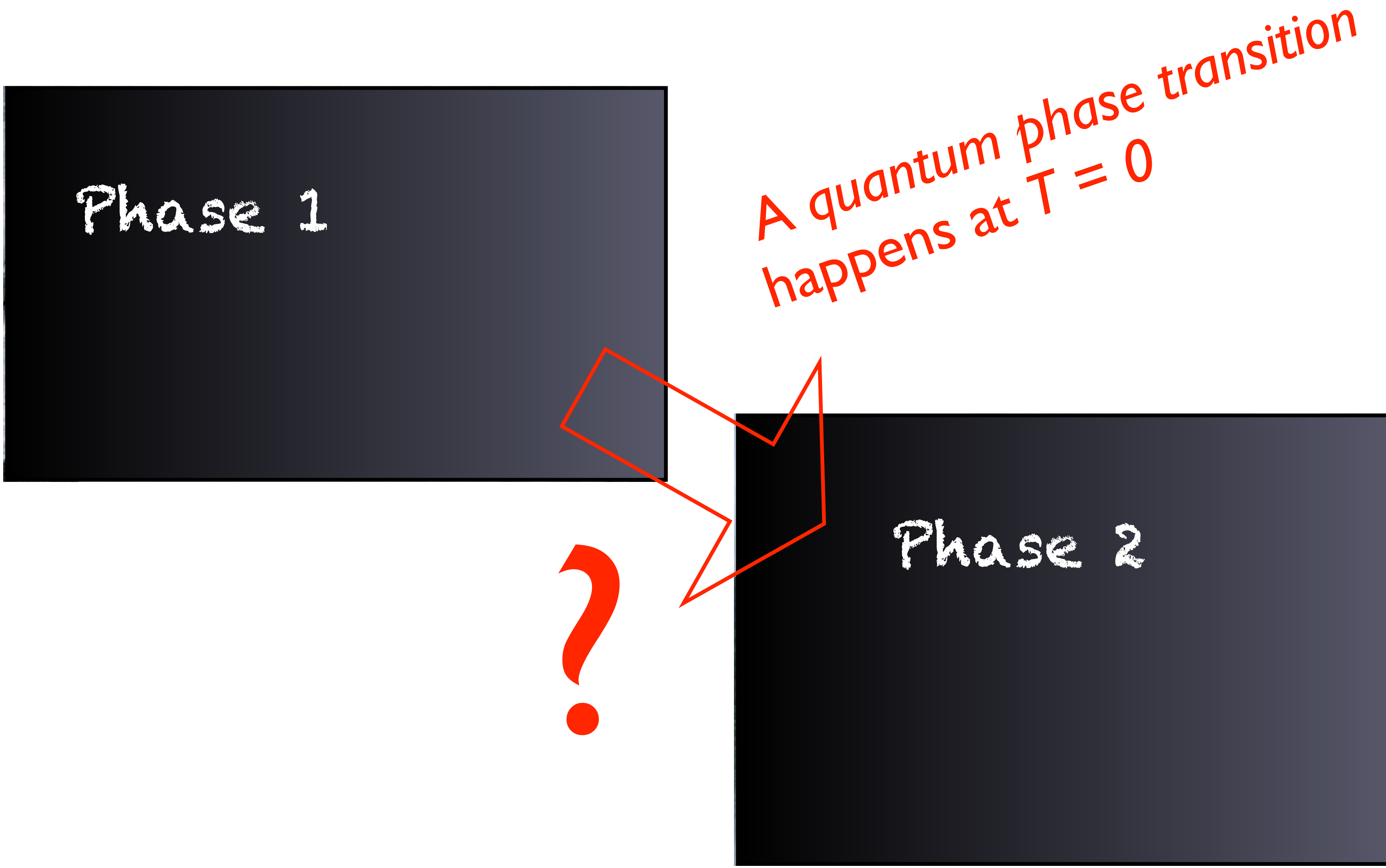


Nature tries to minimize the free energy $F = E -$ ~~TS~~

A quantum phase transition happens at $T = 0$



Nature tries to minimize the free energy $F = E - TS$

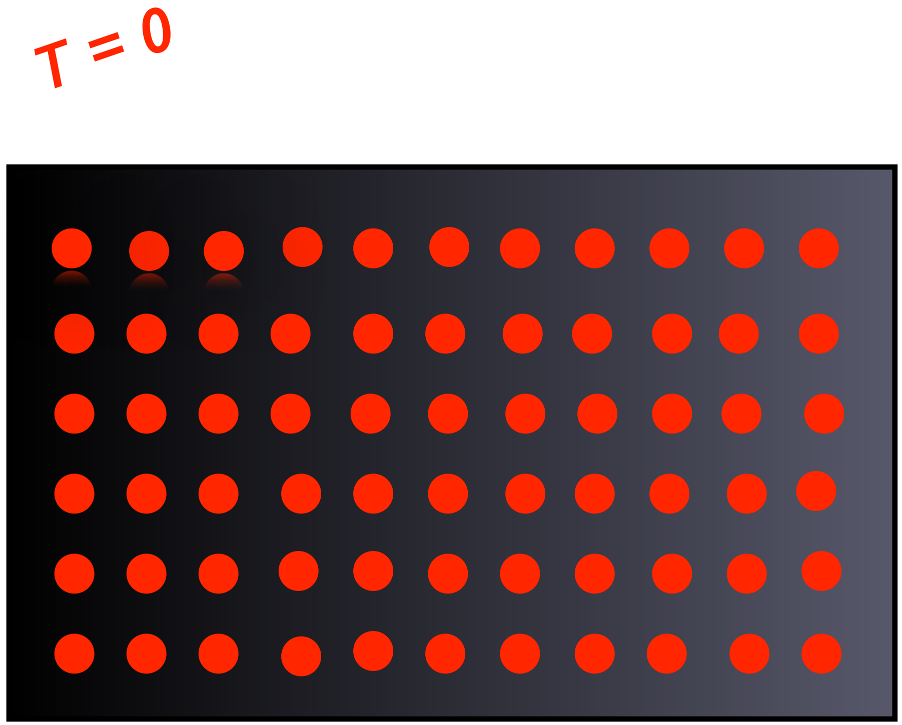


Nature tries to minimize the *energy* E

$T = 0$

consider a bunch of particles with repulsive interactions

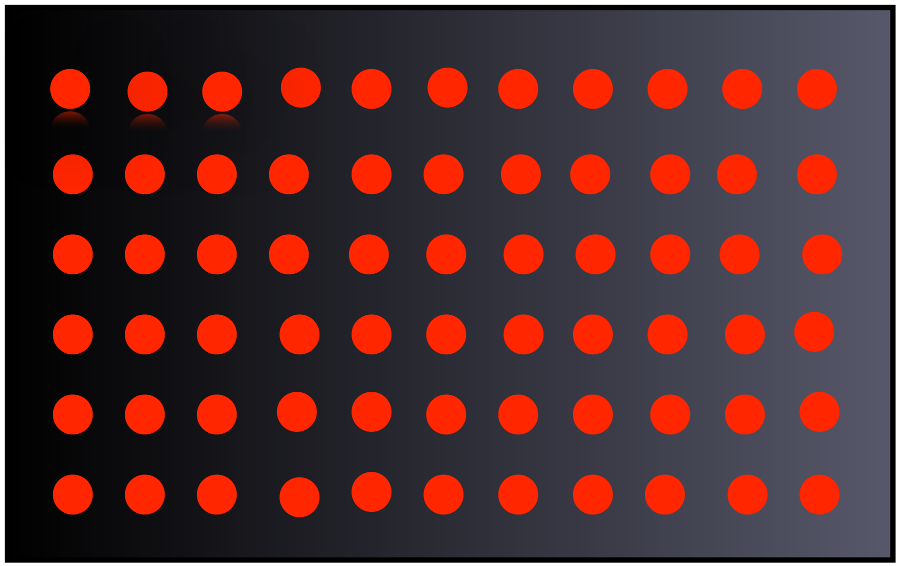
Nature tries to minimize the *energy* E



Lowest energy with all particles at rest, forming a lattice!

Nature tries to minimize the *energy* E

$T = 0$



Lowest energy with all particles at rest, forming a lattice!

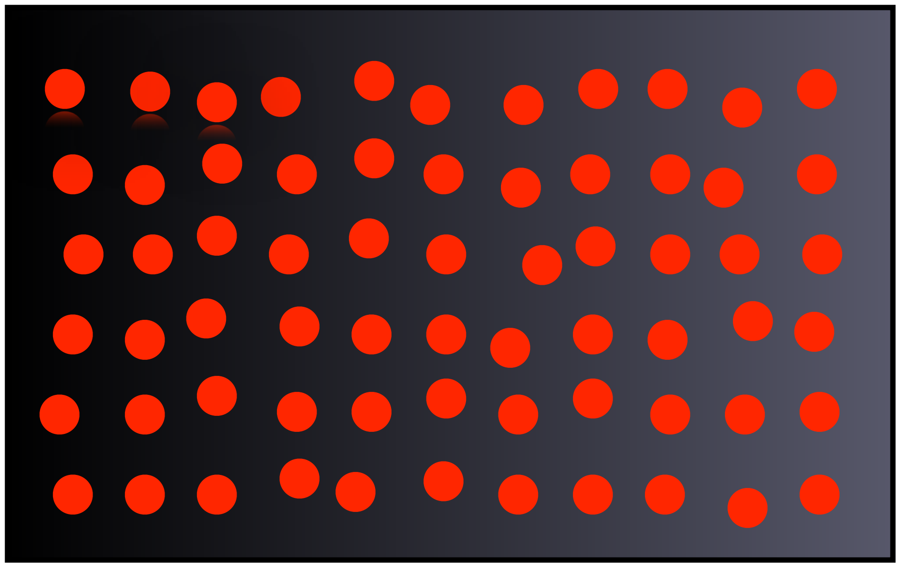
No!

$$\Delta x \, \Delta p \geq \hbar/2\,!$$

W. Heisenberg, 1926

Nature tries to minimize the *energy* E

$T = 0$



snapshot, *with* fluctuations

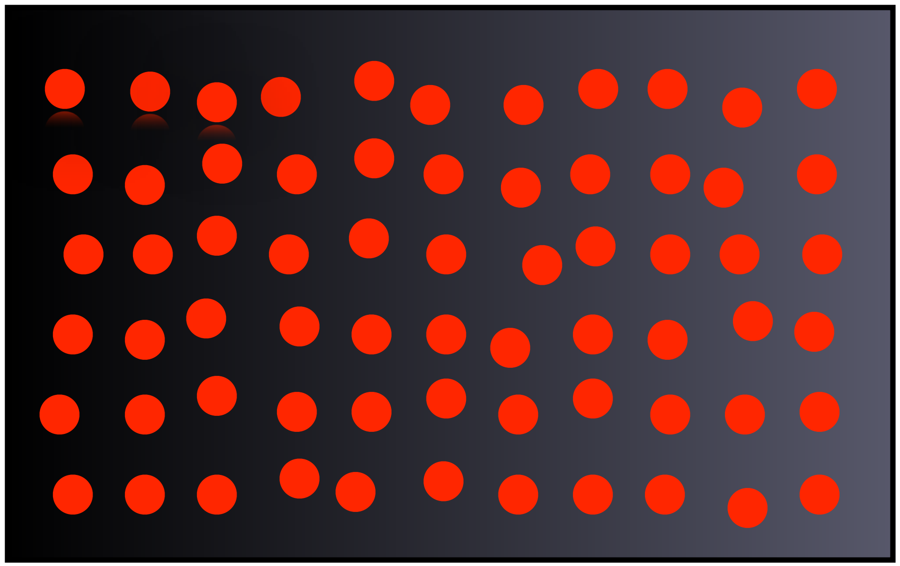
Lowest energy with all particles at rest, forming a lattice!

Quantum fluctuations!

W. Heisenberg, 1926

Nature tries to minimize the *energy* E

$T = 0$



snapshot, *with* fluctuations

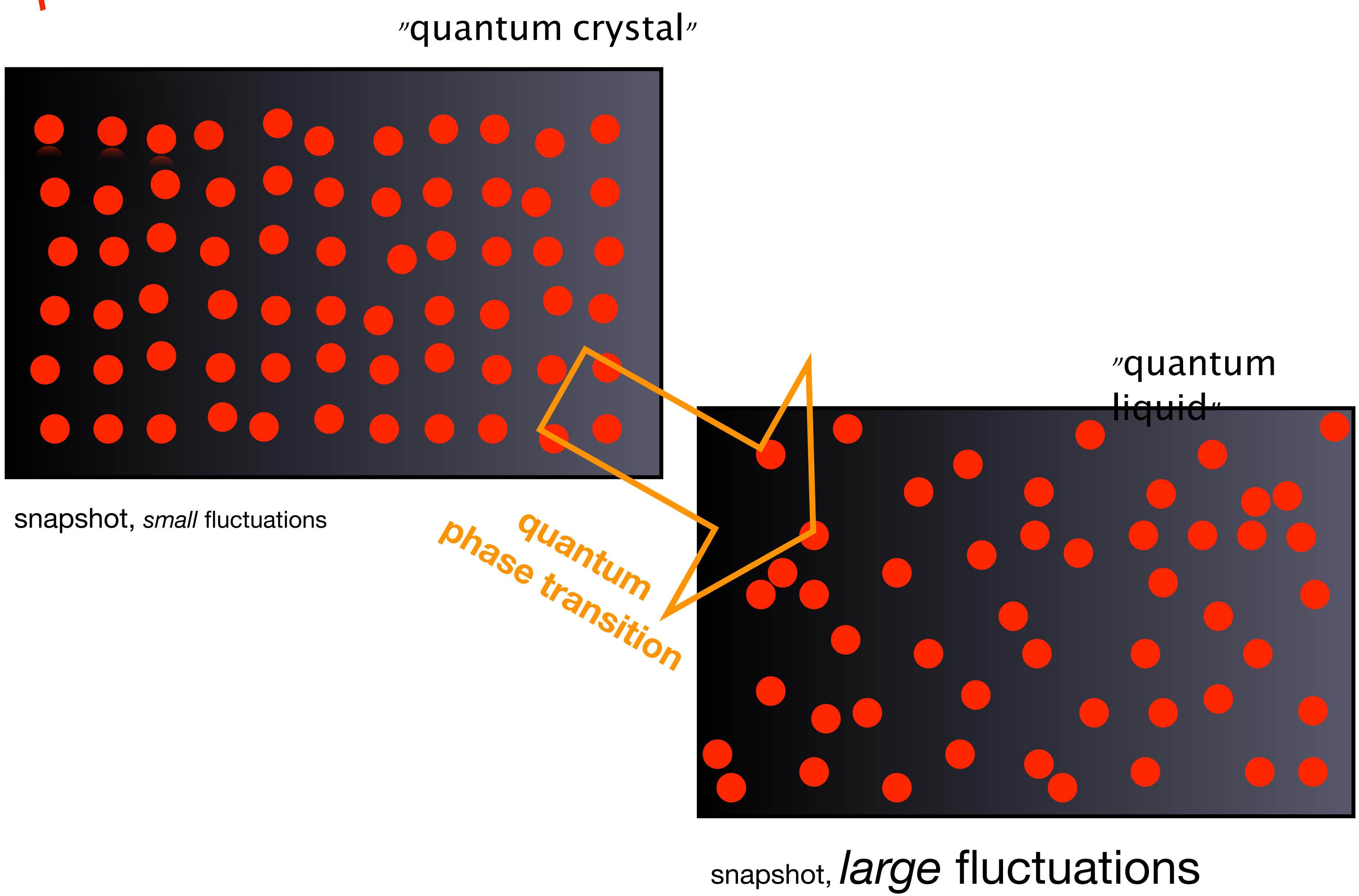
Quantum fluctuations!

W. Heisenberg, 1926

To minimize the energy ***and*** satisfy the *Uncertainty Principle*, the system must strike the ”right” balance between potential and kinetic energies...

Nature tries to minimize the *energy* E

$T = 0$



How to make *large* fluctuations and induce a quantum phase transition?

Example: system of coupled qubits

a degree of freedom that can be in a superposition of two states $|1\rangle, |0\rangle \in \mathbb{C}^2$

$$|\psi\rangle = \alpha\,|1\rangle + \beta\,|0\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

Example: system of coupled qubits

qubit states are realized by spin-$\frac{1}{2}$ states:

$$|\psi\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

eigenstates of $S^z = \frac{\hbar}{2}\sigma^z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

with eigenvalues $\pm\,\frac{\hbar}{2}$

Example: system of coupled qubits

E. Ising, 1925

$$H_{\text{I}sing} = -\sum_j \sigma_j^z \sigma_{j+1}^z$$

qubit states are realized by spin-$\frac{1}{2}$ states:

$|\psi\rangle = \alpha\,|\!\uparrow\rangle + \beta\,|\!\downarrow\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$

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eigenstates of $S^z = \frac{\hbar}{2}\sigma^z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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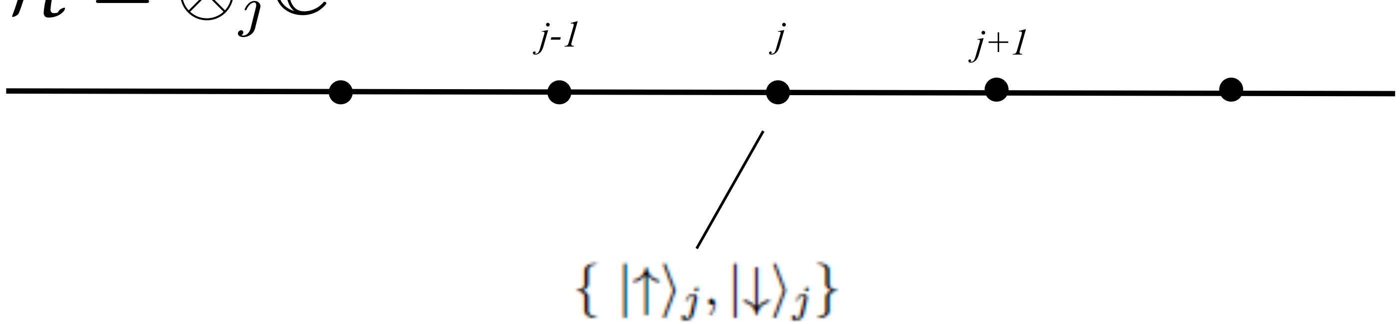
$$\mathcal{H} = \otimes_j \mathbb{C}^2$$



$$H_{\mathrm{I}sing} = -\sum_j \sigma_j^z \sigma_{j+1}^z$$

$$|\,\Psi_\uparrow\,\rangle = \quad \ldots \otimes \; |\!\uparrow\rangle_{j-1} \otimes |\!\uparrow\rangle_j \otimes \; |\!\uparrow\rangle_{j+1} \; \otimes \; \ldots$$

$$|\,\Psi_\downarrow\,\rangle = \quad \ldots \otimes \; |\!\downarrow\rangle_{j-1} \otimes |\!\downarrow\rangle_j \otimes \; |\!\downarrow\rangle_{j+1} \; \otimes \; \ldots$$

Two possible groundstates for $N = 2m, m \in \mathbb{N}$
One of them, say $|\,\Psi_\uparrow\,\rangle$, is selected in the thermodynamic limit by
spontaneous symmetry breaking (from influence of boundaries, perturbations,...)

Note: *without* symmetry breaking we could have formed a superposition of
macroscopic spin-up and spin-down states, a *"Schrödinger cat state"!*

$$H_{\mathrm{I}sing} = -\sum_j \sigma_j^z \sigma_{j+1}^z$$

$$| \, \Psi_{\uparrow} \rangle = \quad ... \otimes \; |\uparrow\rangle_{j-1} \otimes |\uparrow\rangle_j \otimes \; |\uparrow\rangle_{j+1} \; \otimes \; ...$$

$$H_{\substack{quantum \\ Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$

P. Pfeuty, Ann. Phys. **57**, 90 (1970)

$$H_{\substack{quantum\\Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$

P. Pfeuty, Ann. Phys. **57**, 90 (1970)

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generators of SU(2)

$$H_{\substack{quantum \\ Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenstates $|\rightleftarrows\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$

with eigenvalues ± 1

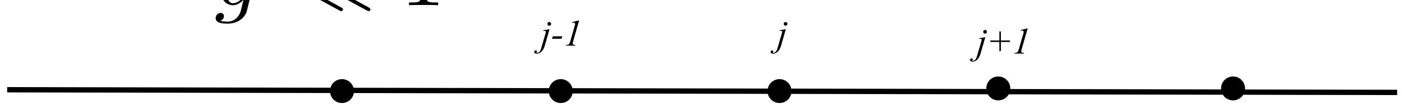
"But you can't go through life applying Heisenberg's Uncertainty Principle to everything."

$$H_{\substack{quantum\\Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$

$$g \gg 1$$

$$|\,\Psi_{\rightarrow}\,\rangle = \quad \ldots \otimes \; |{\rightarrow}\rangle_{j-1} \otimes |{\rightarrow}\rangle_j \otimes \; |{\rightarrow}\rangle_{j+1} \; \otimes \; \ldots$$

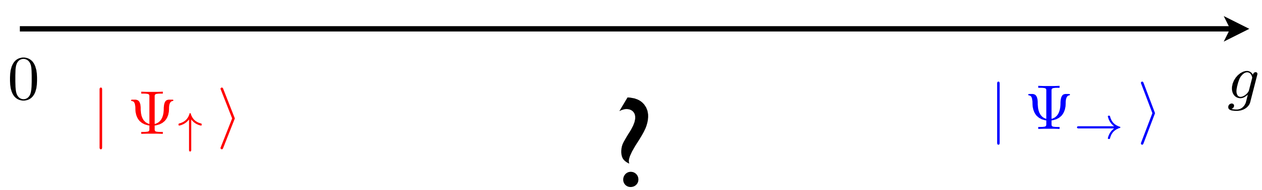
$$g \ll 1$$



$$|\,\Psi_{\uparrow}\,\rangle = \quad \ldots \otimes \; |{\uparrow}\rangle_{j-1} \otimes |{\uparrow}\rangle_j \otimes \; |{\uparrow}\rangle_{j+1} \; \otimes \; \ldots$$

with spontaneous symmetry breaking

$$H_{\substack{quantum\\Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g\sum_j \sigma_j^x$$



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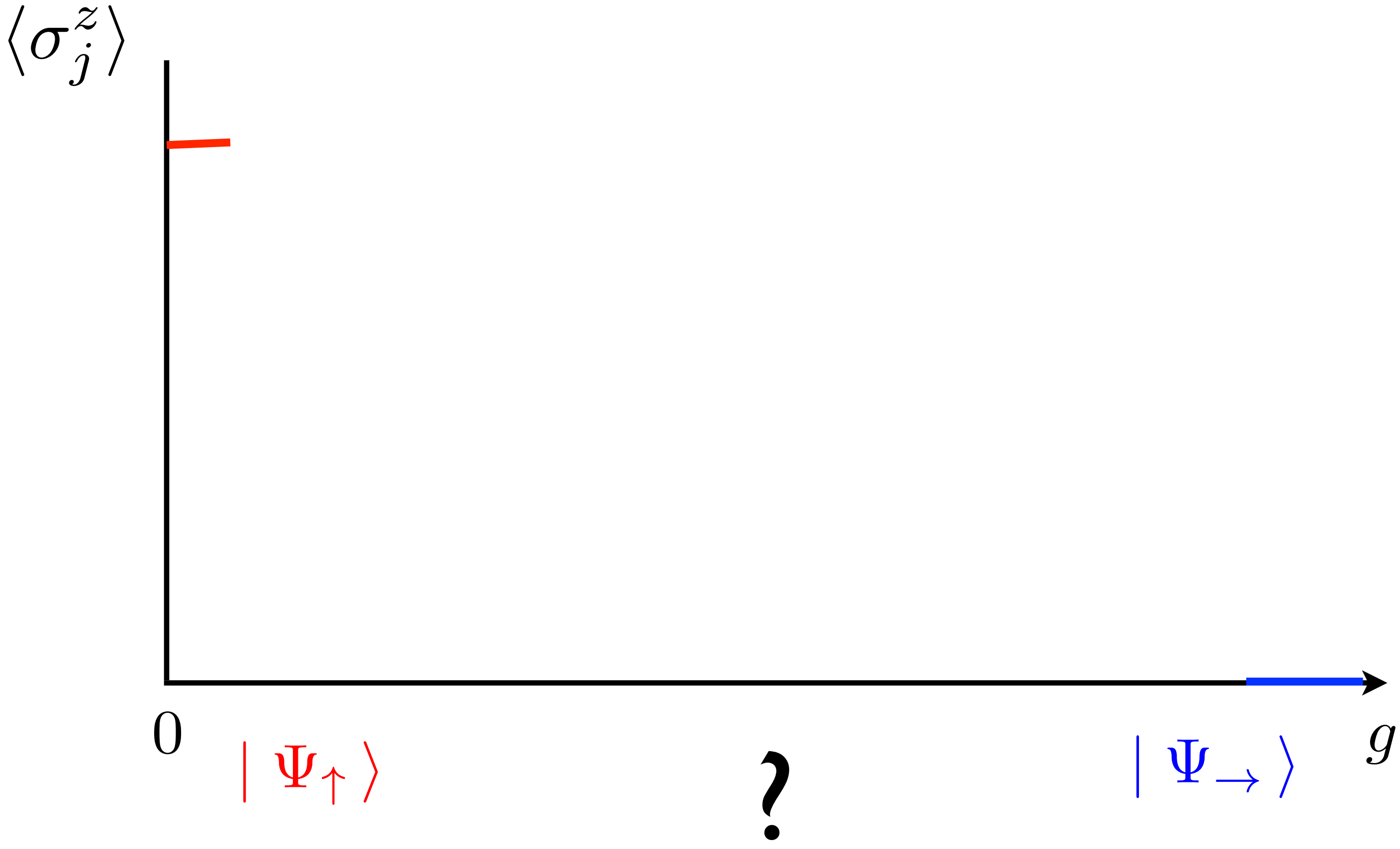

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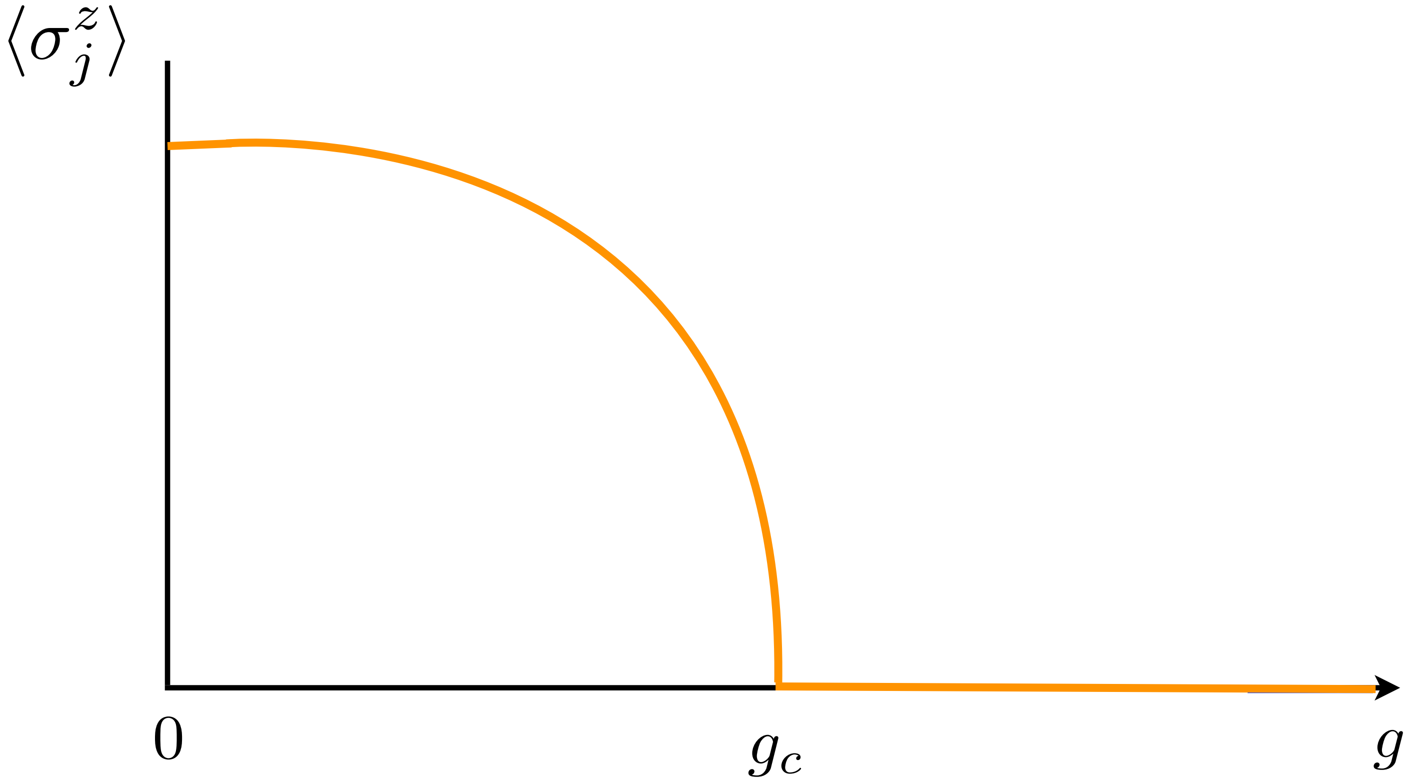

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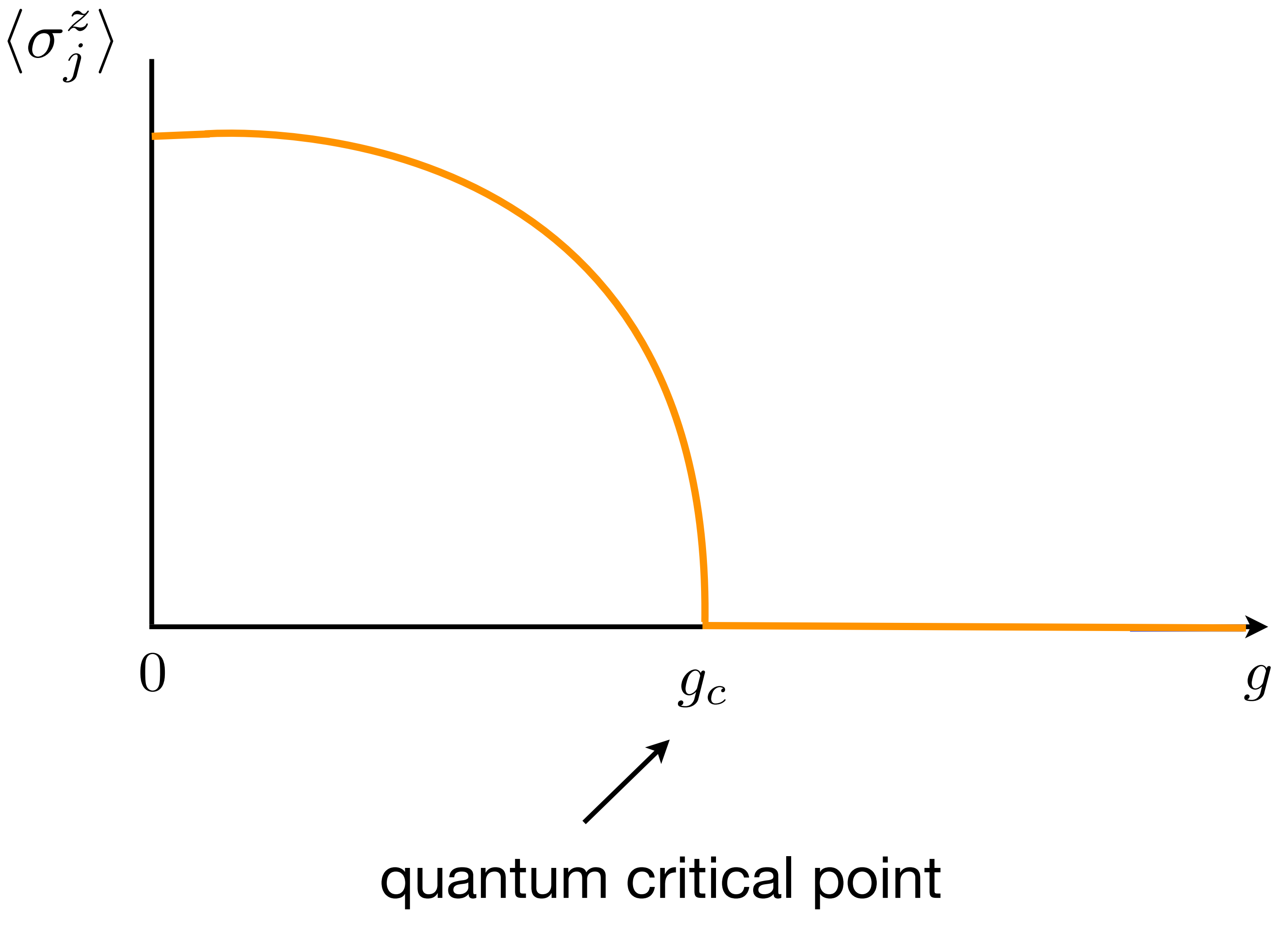

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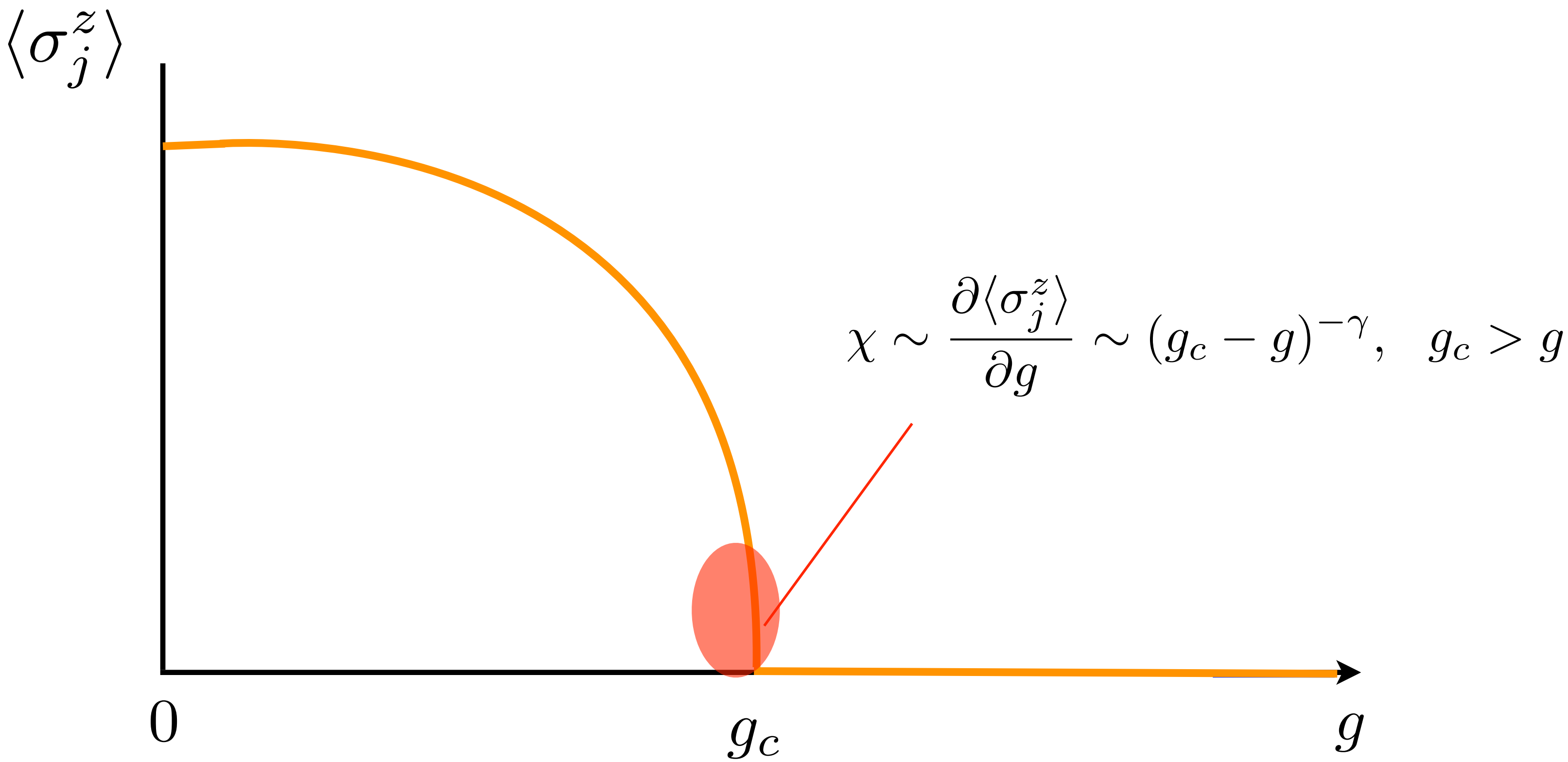

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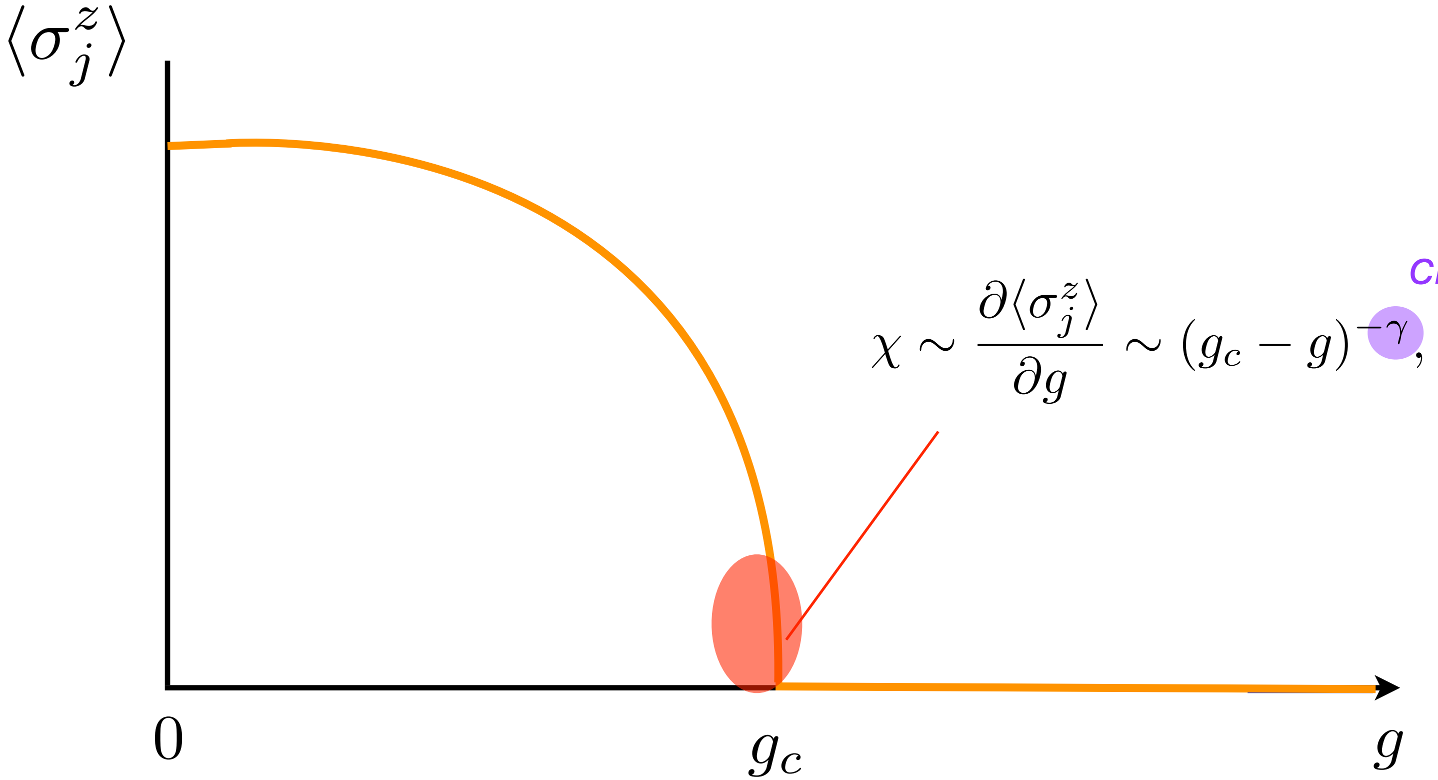
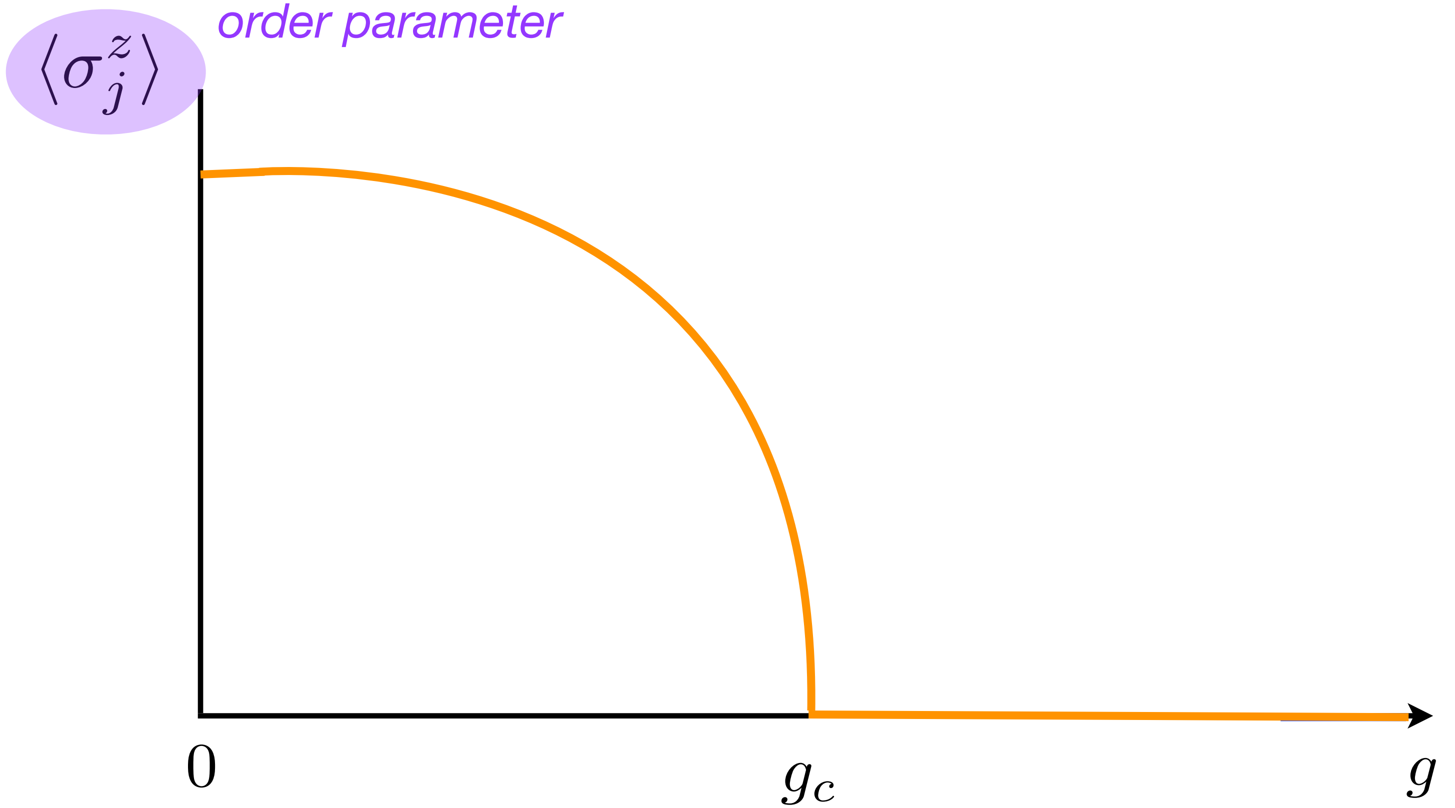


critical exponent

$$\chi \sim \frac{\partial \langle \sigma_j^z \rangle}{\partial g} \sim (g_c - g)^{-\gamma}, \quad g_c > g$$

$$H_{\substack{quantum \\ Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$



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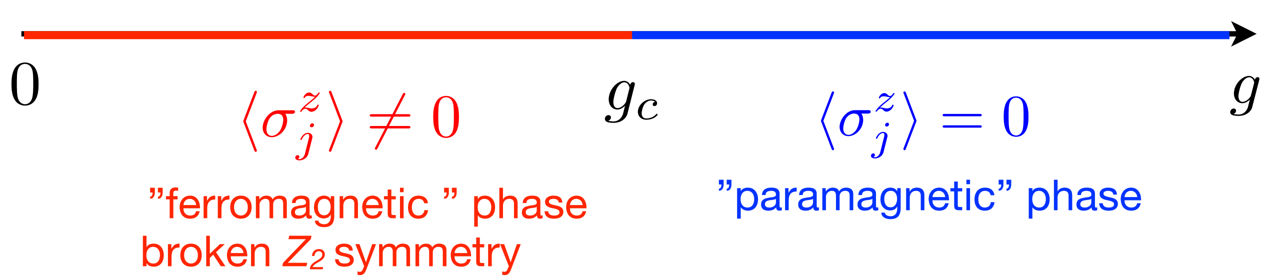
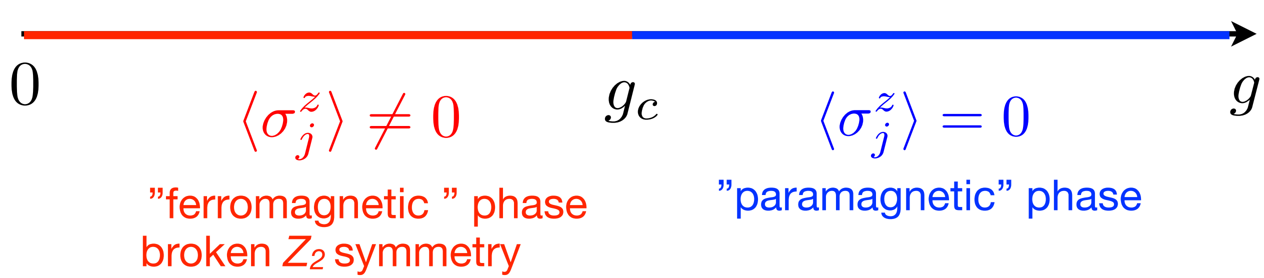


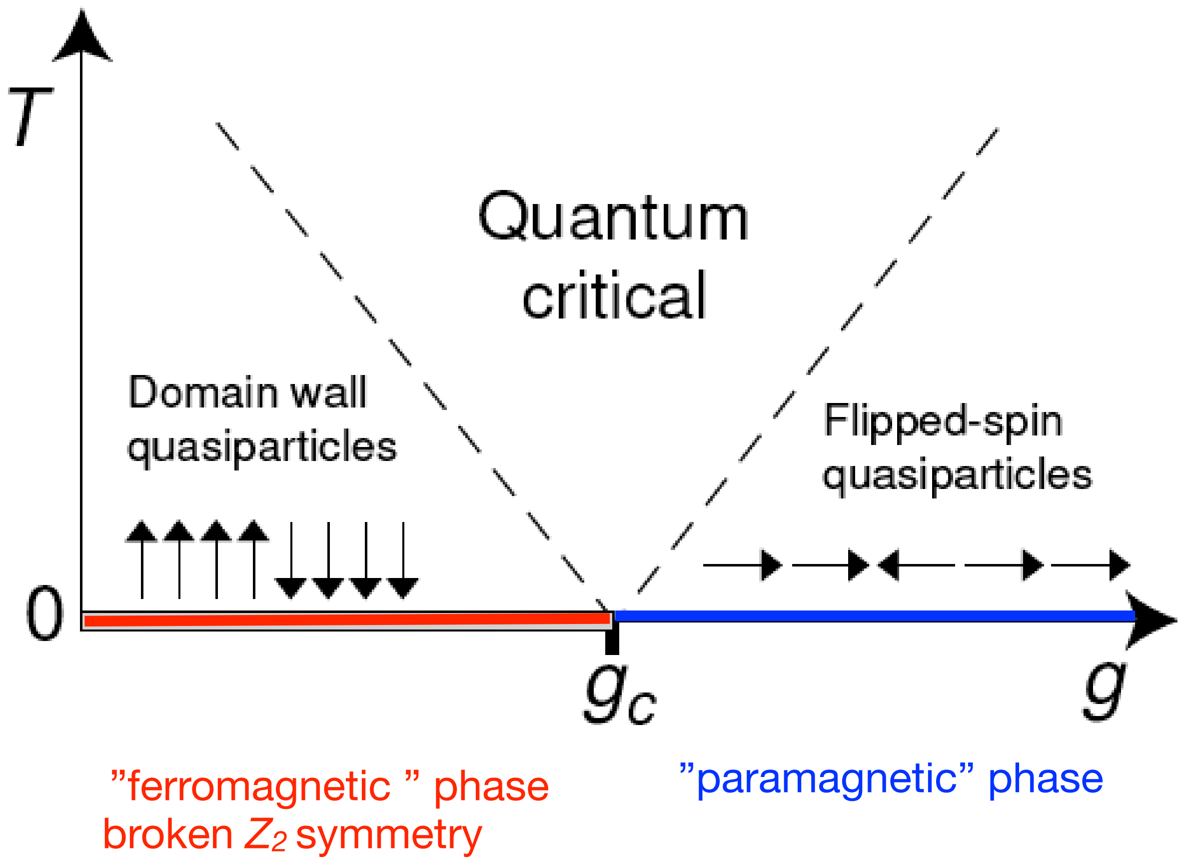
$$H_{\substack{quantum \\ Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$

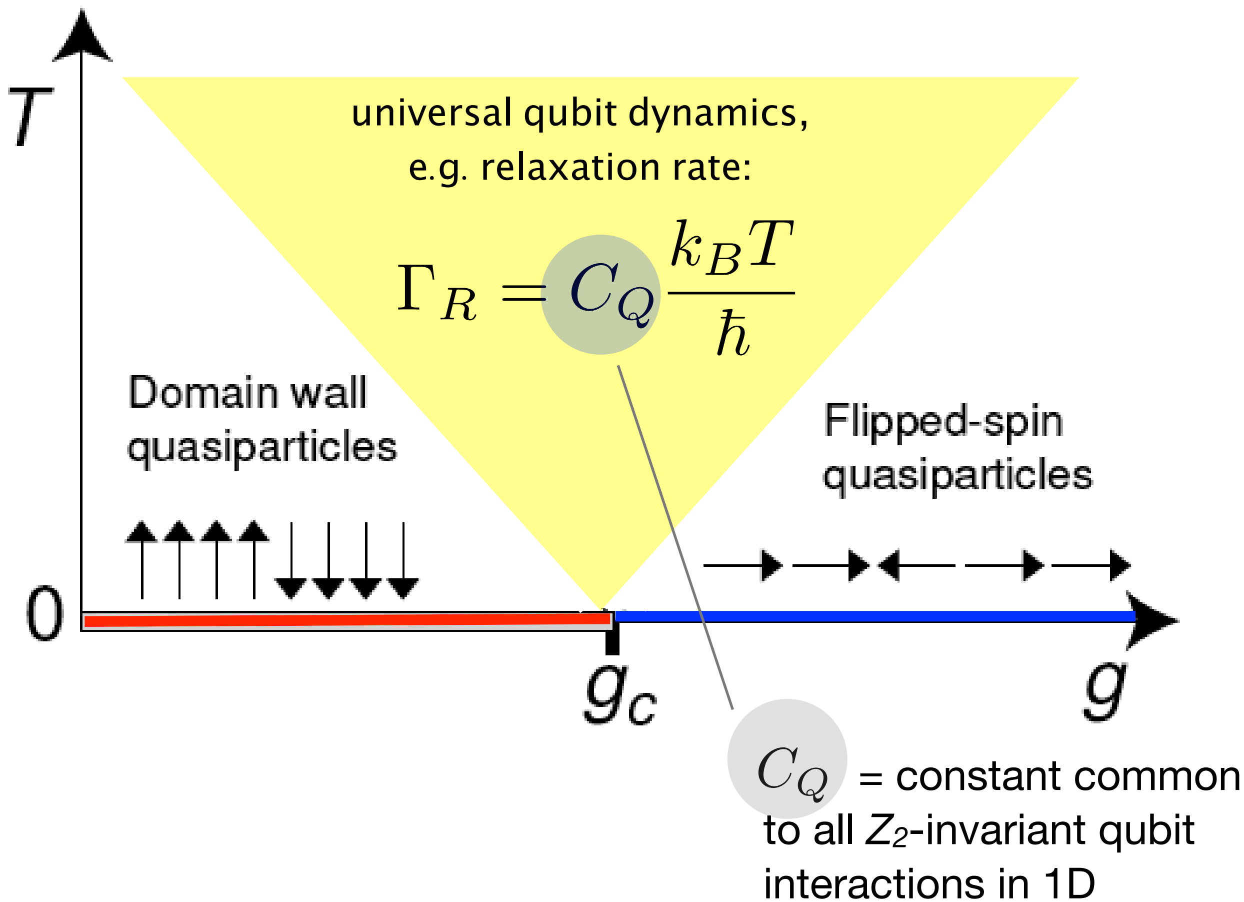
Why bother?
Who cares about a phase transition at T = 0?



$$H_{\substack{quantum\\Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$

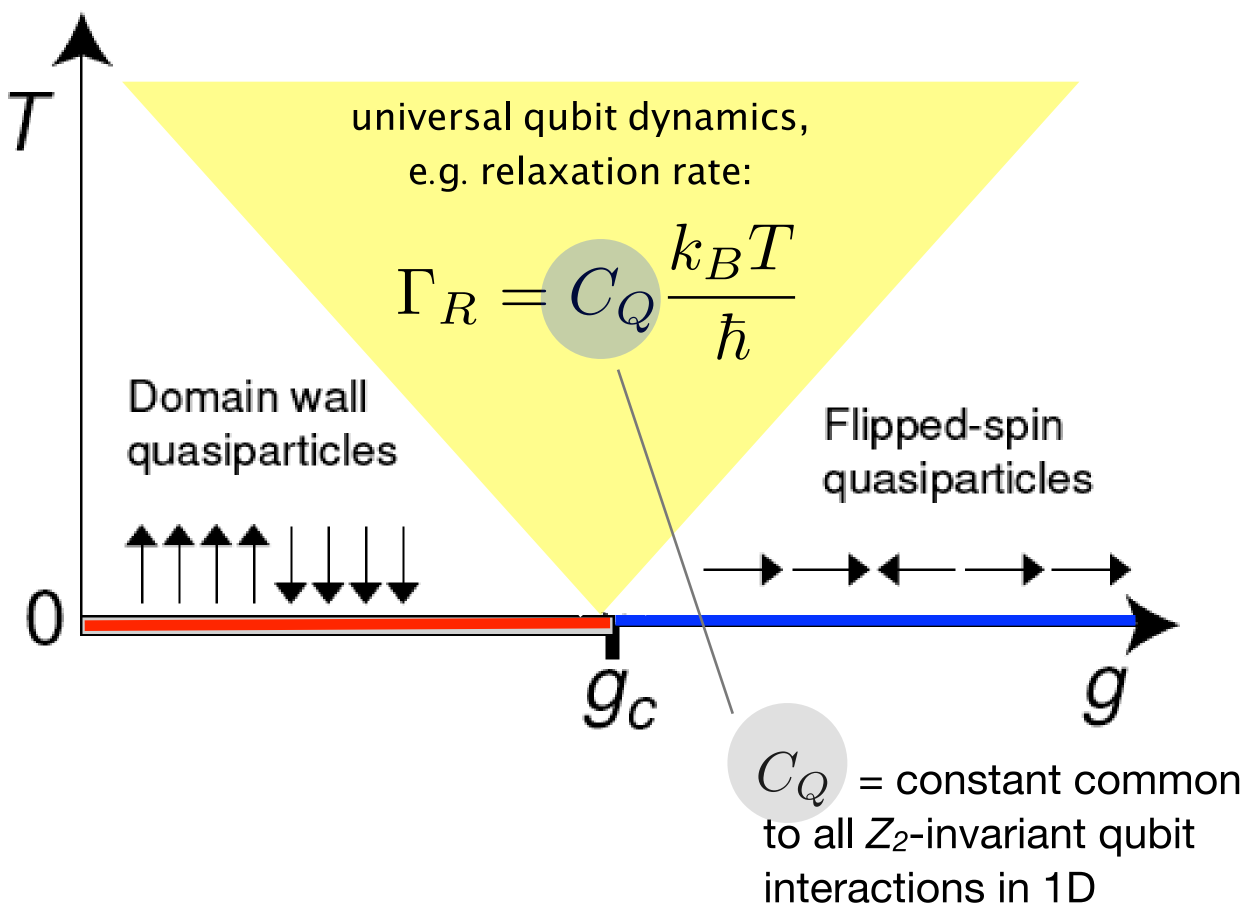
Quantum phase transitions control the physics at finite T!

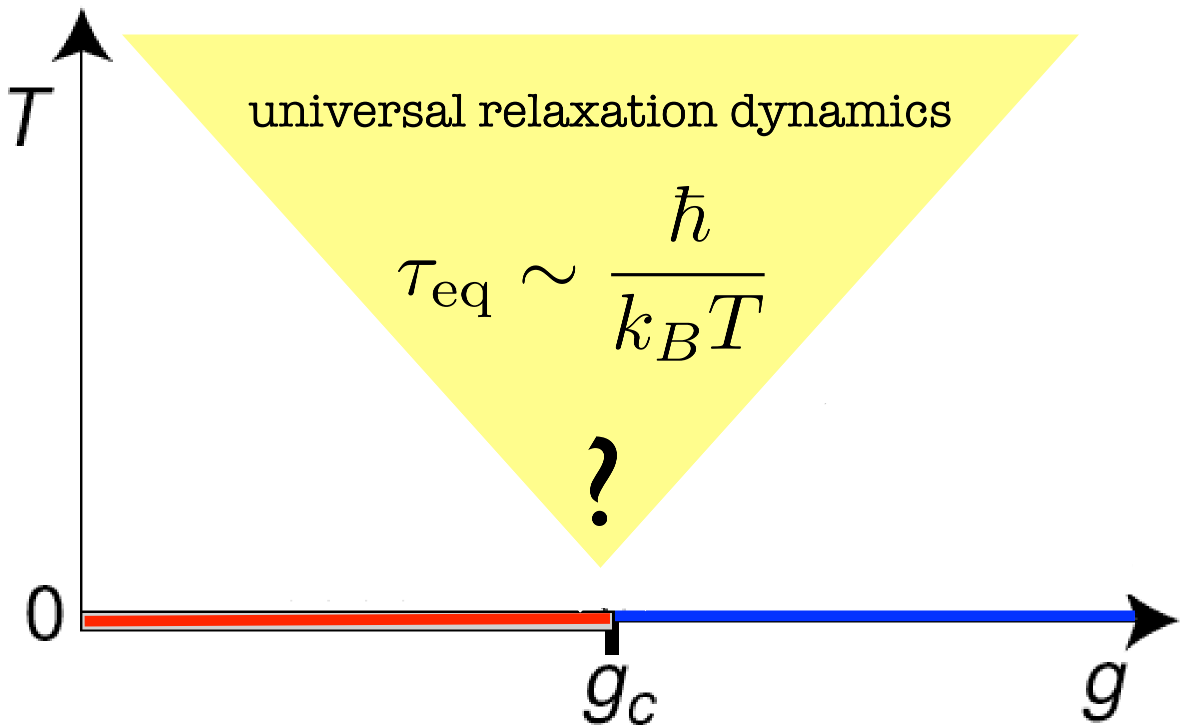




T
universal qubit dynamics,
e.g. relaxation rate:
$\Gamma_R = C_Q \frac{k_B T}{\hbar}$
Domain wall
quasiparticles
Flipped-spin
quasiparticles
0
g_c
g
C_Q = constant common
to all Z_2-invariant qubit
interactions in 1D

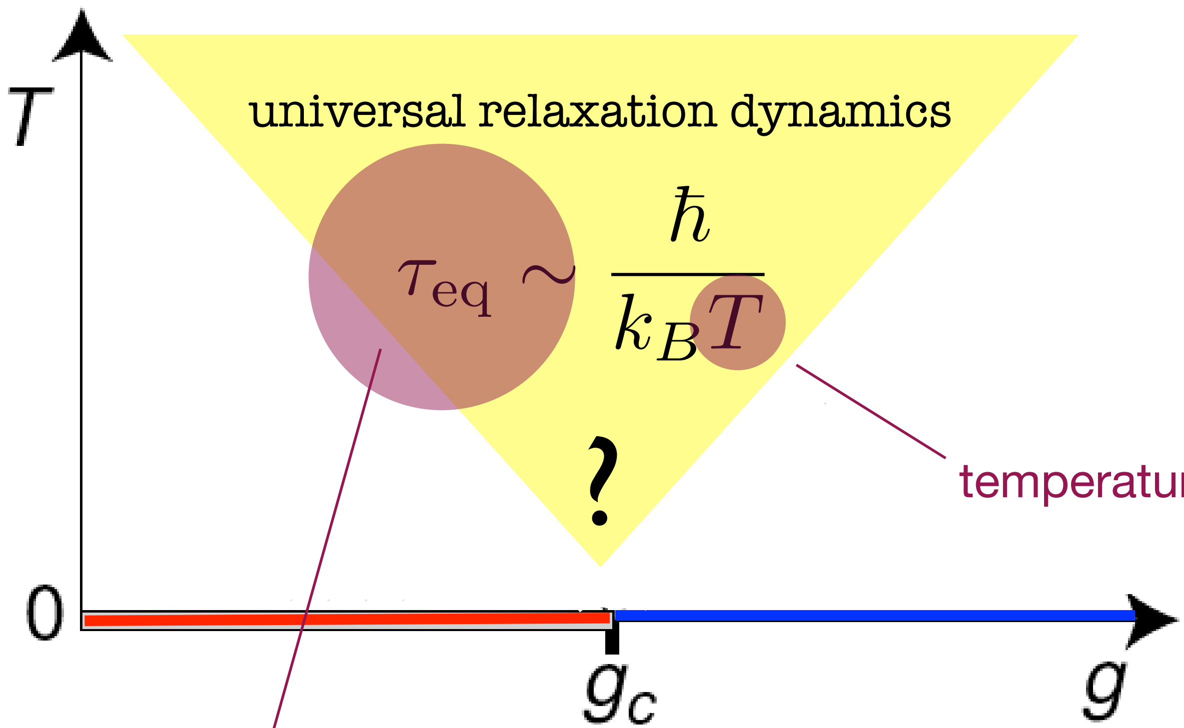
A collective dynamical property of coupled qubits is determined only by the absolute temperature and by fundamental constants of Nature !




T
universal relaxation dynamics
$\tau_{\rm eq} \sim \frac{\hbar}{k_B T}$
?
0
g_c
g

”AdS/CFT correspondence”

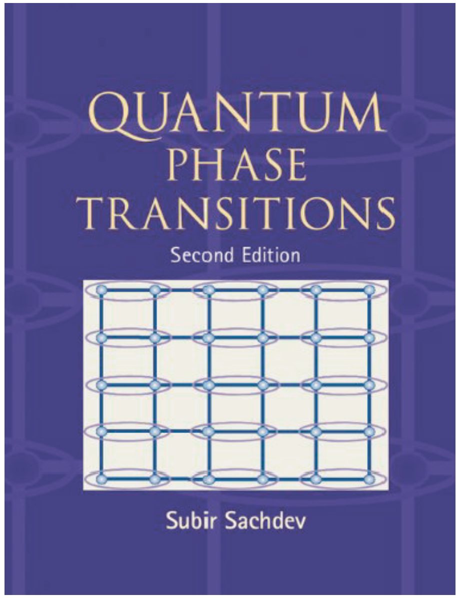
review: S. Sachdev & B. Keimer, *Physics Today, 29, Feb 2011*



temperature of Hawking radiation

damping time of normal modes modes
of gravitational field around a black hole

For more about the basic theory of quantum phase transitions, see the book by Subir Sachdev:



What about the connection to integrable systems and quantum information?

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$$H_{\substack{quantum\\ Ising}} = -\sum_j \sigma_j^z \sigma_{j+1}^z - g \sum_j \sigma_j^x$$

Jordan-Wigner transformation

$$H_{fermion} = -\sum_j (a_j^\dagger a_{j+1} + a_j^\dagger a_{j+1}^\dagger - g a_j^\dagger a_j + \text{h.c.})$$

Exactly solvable! This is the reason why the model has become a paradigm for QPTs... however, from the perspective of the theory of integrable models, it's a bit too simple... no Yang-Baxter algebra, no quantum group extensions,....

What about the connection to integrable systems and quantum information?

What about the connection to integrable systems and quantum information?

It all started back in 2002...

PHYSICAL REVIEW A, **66**, 032110 (2002)

Entanglement in a simple quantum phase transition

Tobias J. Osborne[1,2,*] and Michael A. Nielsen[2,†]
[1]*Department of Mathematics, University of Queensland 4072, Brisbane, Queensland, Australia*
[2]*Centre for Quantum Computer Technology and Department of Physics, University of Queensland 4072, Brisbane, Queensland, Australia*
(Received 27 February 2002; published 23 September 2002)

What entanglement is present in naturally occurring physical systems at thermal equilibrium? Most such systems are intractable and it is desirable to study simple but realistic systems that can be solved. An example of such a system is the one-dimensional infinite-lattice anisotropic XY model. This model is exactly solvable using the Jordan-Wigner transform, and it is possible to calculate the two-site reduced density matrix for all pairs of sites. Using the two-site density matrix, the entanglement of formation between any two sites is calculated for all parameter values and temperatures. We also study the entanglement in the transverse Ising model, a special case of the XY model, which exhibits a quantum phase transition. It is found that the next-nearest-neighbor entanglement (though not the nearest-neighbor entanglement) is a maximum at the critical point. Furthermore, we show that the critical point in the transverse Ising model corresponds to a transition in the behavior of the entanglement between a single site and the remainder of the lattice.

DOI: 10.1103/PhysRevA.66.032110 PACS number(s): 03.65.Ud, 73.43.Nq, 05.50.+q

letters to nature

Scaling of entanglement close to a quantum phase transition

A. Osterloh*†, Luigi Amico*†, G. Falci*† & Rosario Fazio†‡

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† *NEST-INFM, Piazza dei Cavaliezi 7, I-56126 Pisa, Italy*
‡ *Scuola Normale Superiore, Piazza dei Cavaliezi 7, I-56126 Pisa, Italy*

Classical phase transitions occur when a physical system reaches a state below a critical temperature characterized by macroscopic order[1]. Quantum phase transitions occur at absolute zero; they are induced by the change of an external parameter or coupling constant[2], and are driven by quantum fluctuations. Examples include transitions in quantum Hall systems[3], localization in Si-MOSFETs (metal oxide silicon field-effect transistors; ref. 4) and the superconductor–insulator transition in two-dimensional systems[5,6]. Both classical and quantum critical points are governed by a diverging correlation length, although quantum systems possess additional correlations that do not have a classical counterpart. This phenomenon, known as entanglement, is the resource that enables quantum computation and communication[8]. The role of entanglement at a phase transition is not captured by statistical mechanics—a complete classification of the critical many-body state requires the introduction of concepts from quantum information theory[9]. Here we connect the theory of critical phenomena with quantum information by exploring the entangling resources of a system close to its quantum critical point. We demonstrate, for a class of one-dimensional magnetic systems, that entanglement shows scaling behaviour in the vicinity of the transition point.

NATURE **416**, 608 (2002)

What about the connection to integrable systems and quantum information?

"...to pinpoint what is genuinely quantum in a zero-temperature phase transition requires the introduction of concepts from quantum information theory..."

" ... close to the quantum critical point the entanglement depends strongly on the field... so it could be tuned, realizing an "entanglement switch."

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NATURE **416**, 608 (2002)

What about the connection to integrable systems and quantum information?

"We have argued that the quantum critical point of a lattice system corresponds to the situation where the lattice is maximally **entangled**. Evidence for this conjecture was found in our *single-site* ***entanglement*** results for the groundstate of the quantum Ising model."

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Classical phase transitions occur when a physical system reaches a state below a critical temperature characterized by macroscopic order[1]. Quantum phase transitions occur at absolute zero; they are induced by the change of an external parameter or coupling constant[2], and are driven by quantum fluctuations. Examples include transitions in quantum Hall systems[3], localization in Si-MOSFETs (metal oxide silicon field-effect transistors; ref. 4) and the superconductor–insulator transition in two-dimensional systems[5,6]. Both classical and quantum critical points are governed by a diverging correlation length, although quantum systems possess additional correlations that do not have a classical counterpart. This phenomenon, known as entanglement, is the resource that enables quantum computation and communication[8]. The role of entanglement at a phase transition is not captured by statistical mechanics—a complete classification of the critical many-body state requires the introduction of concepts from quantum information theory[9]. Here we connect the theory of critical phenomena with quantum information by exploring the entangling resources of a system close to its quantum critical point. We demonstrate, for a class of one-dimensional magnetic systems, that entanglement shows scaling behaviour in the vicinity of the transition point.

NATURE **416**, 608 (2002)

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Osborne and Nielsen, 2002

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Adiabatic quantum computation and quantum phase transitions

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We analyze the ground-state entanglement in a quantum adiabatic evolution algorithm designed to solve the *NP*-complete Exact Cover problem. The entropy of entanglement seems to obey linear and universal scaling at the point where the energy gap becomes small, suggesting that the system passes near a quantum phase transition. Such a large scaling of entanglement suggests that the effective connectivity of the system diverges as the number of qubits goes to infinity and that this algorithm cannot be efficiently simulated by classical means. On the other hand, entanglement in Grover's algorithm is bounded by a constant.

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"Adiabatic quantum computation inherently brings the quantum system near to a point where a quantum phase transtion takes place. Entanglement is then expected to pervade the system..."

The 2002 papers by Osborne & Nielsen and Osterloh *et al.* inspired an enormous (and still ongoing!) research effort on entanglement in quantum many-particle systems...

- theory of topological quantum matter
- tensor network methods for numerics
- **theory of quantum phase transitions**
- ....

For reviews, see L. Amico et al., Rev. Mod. Phys. **80**, 517 (2008); X.-G. Wen, arXiv:1210.1281 [topological order]; N. Schuch, arXiv:1306.5551 [tensor networks]

Outline of lectures

1. What is entanglement? How to calculate it?

2. Quantum phase transitions in the 1D Hubbard model

3. Integrability of the 1D Hubbard model, Bethe Ansatz, and all that...

4. Entanglement from the Bethe Ansatz: What do we learn about quantum phase transitions in the 1D Hubbard model?

5. Summary and outlook

The connection between entanglement and quantum phase transitions is rather more intriguing than suggested in the original 2002 papers on the subject!