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Non-Fermi liquids

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Our present understanding of how the interactions between electrons affect the metallic state has, for forty years, rested on the foundations of Landau's Fermi-liquid theory. It provides the basis for understanding metals in terms of weakly interacting electron (-like) particles. Recent years have seen the discovery of metals which appear to fall outside this framework—perhaps most notably in the normal state of the high temperature cuprate superconductors. While the theory for understanding the cuprate metals remains controversial, there are a number of clear examples where we do believe we understand the new underlying theoretical concepts. In this article I illustrate four such routes towards forming a non-Fermi liquid metal and illustrate, where possible, how these have been realized in a number of materials. The proximity to a quantum phase transition and reduced effective dimensionality can both play important roles.

1. Introduction

Condensed matter physics is a subject continually inspired by the fabrication of new materials. With each new generation of materials synthesized comes a new set of challenges for the condensed matter physics community to understand and exploit. To the observer this may seem surprising since the basic interactions governing the motion of the electrons and atomic nuclei in a solid have long been known. While this is true, with each new compound we see these basic forces at work in a different local environment and the result is rarely a trivial extrapolation of the physics we knew before. Instead, with each level of complexity we see new types of phenomena arising—every bit as fundamental as the bare interactions with which we began (see Anderson 1972). Examples of such radically new behaviour include the appearance of fractionally charged objects in the fractional quantum Hall effect, the observation of super-massive electrons in so-called heavy fermion materials and the possibility of the electron decaying into new types of particle in certain one-dimensional materials. One of the current areas of excitement in the field has been motivated by the discovery of certain metallic compounds which seem to fall outside of the framework of our current theory of metals.

It is hard to imagine describing the physics of metals without beginning with the electron yet, remarkably, over the past decade there is a growing field of condensed matter physics devoted to understanding metals where the electron seems to be precisely the wrong place to start. Of course we are well aware that the basic ingredients of solids are atoms with their valence and core electrons and nuclei but, as often happens in condensed matter, in bringing such atoms together what emerges can be very different from the constituent parts, with the possibility of completely new types of 'particles'. (Two more familiar examples of new particles appearing in condensed matter systems are phonons—quantized lattice vibrations—and magnons—waves of spin in a magnet.) Yet for the understanding of the metallic state the electron has remained the unrivaled basis since Drude's initial work at the beginning of this century (Drude 1900). The success of the single electron picture of metals rests on Landau's seminal work in the 1950s developing Fermi-liquid theory (Landau 1956, 1957, 1958). Yet as this century closes we are seeing the discovery of materials, including the cuprate superconductors and other oxides, which seem to lie outside this framework. In this review I will attempt to give a flavour of our attempts to understand such 'non-Fermi liquid' metals. Many of the theoretical ideas have been developed using rather complex mathematical machinery which I make no apology for omitting in favour of a more descriptive approach. There are however a number of results which can be obtained relatively simply using Fermi's golden rule (together with Maxwell's equations) and I have included these for readers who would like to see where some of the properties are coming from.

The outline of this review is as follows. I begin with a description of Fermi-liquid theory itself. This theory tells us why one gets a very good description of a metal by treating

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it as a gas of Fermi particles (i.e. that obey Pauli's exclusion principle) where the interactions are weak and relatively unimportant. The reason is that the particles one is really describing are not the original electrons but electron-like quasiparticles that emerge from the interacting gas of electrons. Despite its recent failures which motivate the subject of non-Fermi liquids, it is a remarkably successful theory at describing many metals including some, like UPt_3, where the interactions between the original electrons are very important. However, it is seen to fail in other materials and these are not just exceptions to a general rule but are some of the most interesting materials known. As an example I discuss its failure in the metallic state of the high temperature superconductors.

I then present four examples which, from a theoretical perspective, generate non-Fermi liquid metals. These all show physical properties which can not be understood in terms of weakly interacting electron-like objects:

- *Metals close to a quantum critical point.* When a phase transition happens at temperatures close to absolute zero, the quasiparticles scatter so strongly that they cease to behave in the way that Fermi-liquid theory would predict.
- *Metals in one dimension—the Luttinger liquid.* In one-dimensional metals, electrons are unstable and decay into two separate particles (spinons and holons) that carry the electron's spin and charge respectively.
- *Two-channel Kondo models.* When two independent electrons can scatter from a magnetic impurity it leaves behind 'half an electron'.
- *Disordered Kondo models.* Here the scattering from disordered magnetic impurities is too strong to allow the Fermi quasiparticles to form.

While some of these ideas have been used to try and understand the high temperature superconductors, I will show that in many cases one can see the physics illustrated by these examples in other materials. I believe that we are just seeing the tip of an iceberg of new types of metal which will require a rather different starting point from the simple electron picture to understand their physical properties.

2. Fermi-liquid theory: the electron quasiparticle

The need for a Fermi-liquid theory dates from the first applications of quantum mechanics to the metallic state. There were two key problems. Classically each electron should contribute $3k_B/2$ to the specific heat capacity of a metal—far more than is actually seen experimentally. In addition, as soon as it was realized that the electron had a magnetic moment, there was the puzzle of the magnetic susceptibility which did not show the expected Curie temperature dependence for free moments: $\chi \sim 1/T$.

These puzzles were unravelled at a stroke when Pauli (Pauli 1927, Sommerfeld 1928) (apparently reluctantly—see Hermann *et al.* 1979) adopted Fermi statistics for the electron and in particular enforced the exclusion principle which now carries his name: no two electrons can occupy the same quantum state. In the absence of interactions one finds the lowest energy state of a gas of free electrons by minimizing the kinetic energy subject to Pauli's constraint. The resulting ground state consists of a filled Fermi sea of occupied states in momentum space with a sharp demarcation at the Fermi energy ε_F and momentum $p_F = \hbar k_F$ (the Fermi surface) between these states and the higher energy unoccupied states above. The low energy excited states are obtained simply by promoting electrons from just below the Fermi surface to just above it (see figure 1). They are uniquely labelled by the momentum and spin quantum numbers of the now empty state below the Fermi energy (a hole) and the newly filled state above it. These are known as particle–hole excitations.

This resolves these early puzzles since only a small fraction of the total number of electrons can take part in the processes contributing to the specific heat and magnetic susceptibility. The majority lie so far below the Fermi surface that they are energetically unable to find the unoccupied quantum state required to magnetize them or carry excess heat. Only the electrons within $k_B T$ of the Fermi surface can contribute k_B to the specific heat so the specific heat grows linearly with temperature and is small. Only electrons within $\mu_B B$ of the Fermi surface can magnetize with a moment $\sim \mu_B$ leading to a temperature independent (Pauli) susceptibility. Both quantities are proportional to the density of electron states at the Fermi surface.

These new temperature dependencies exactly matched the experiments both on metals and then later on the fermionic isotope of helium—^{3}He (see, for example, Wheatley 1970). But this in turn raised questions. Why should a theory based on a non-interacting picture work so well in these systems where interactions are undoubtably important? Once interactions are present the problem of finding the low energy states of the electrons becomes much harder. In addition to the kinetic term which favours a low momentum, the energy now contains a potential term which depends on the relative position of all of the electrons. The energy scales of the kinetic energy and Coulomb interaction are comparable at metallic electron densities and, if that were not enough, Heisenberg's uncertainty principle prevents the simultaneous definition of the momentum and the position. How can one proceed and still hope to retain the physics of the non-interacting electron gas which experiment demands?

The answer provided by Landau rests on the concept of 'adiabatic continuity' (Anderson 1981): labels associated with eigenstates are more robust against perturbations than the eigenstates themselves. Consider as an example the

problem of a particle in a box with impenetrable walls illustrated in figure 2. In elementary quantum mechanics one learns that the eigenstates of this problem consist of standing sine waves with nodes at the well walls. The eigenstates of the system can be labelled by the number of additional nodes in the wavefunction with the energy increasing with the number of nodes. Now switch on an additional weak quadratic potential. The new eigenstates of the problem are no longer simple sine waves but involve a mixing of all the eigenstates of the original unperturbed problem. However the number of nodes still remains a good way of labelling the eigenstates of the more complicated problem. This is the essence of adiabatic continuity.

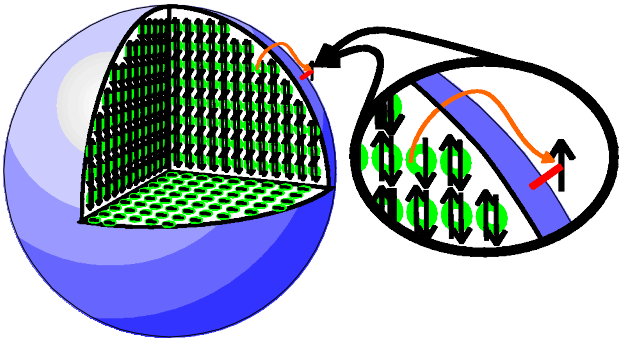


Figure 1. The ground state of the free Fermi gas in momentum space. All the states below the Fermi surface are filled with both a spin-up and a spin-down electron. A particle–hole excitation is made by promoting an electron from a state below the Fermi surface to an empty one above it.

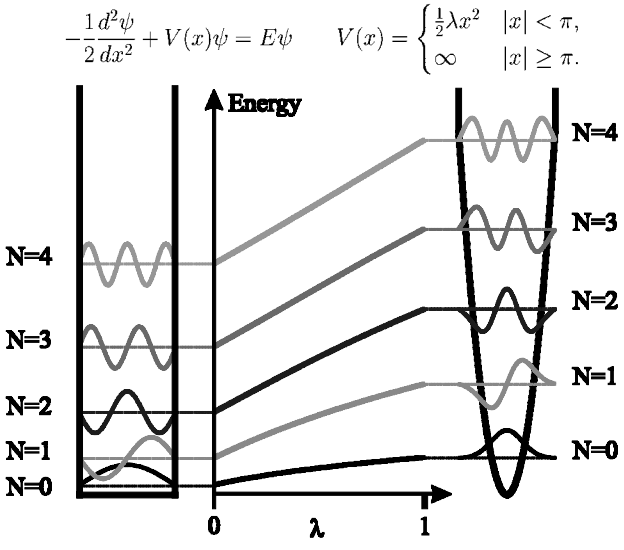


Figure 2. Adiabatic continuity is illustrated in a non-interacting problem by turning on a quadratic potential to a particle confined in box. While the energy levels and the details of the eigenstate wavefunctions evolve subtly, the good quantum numbers of the initial problem (the number of nodes, N, in the wavefunction) are still the appropriate description when the perturbation has been applied.

Landau applied this idea to the interacting gas of electrons. He imagined turning on the interactions between electrons slowly, and observing how the eigenstates of the system evolved. He postulated that there would be a one-to-one mapping of the low energy eigenstates of the interacting electrons with the those of the non-interacting Fermi gas. He supposed that the good quantum numbers associated with the excitations of the non-interacting system would remain good even after the interactions were fully applied. Just as Pauli's exclusion principle determined the allowed labels without the interactions through the presence of a Fermi surface, this feature would remain even with the interactions. We therefore retain the picture of Fermi particles and holes excitations carrying the same quantum numbers as their electron counterparts in the free Fermi gas. These labels are not to be associated with electrons but to 'quasiparticles' to remind us that the wavefunctions and energies are different from the corresponding electron in the non-interacting problem. It is the concept of the fermion quasiparticle that lies at the heart of Fermi-liquid theory. It accounts for the measured temperature dependences of the specific heat and Pauli susceptibility since these properties only require the presence of a well defined Fermi surface, and are not sensitive to whether it is electrons or quasiparticles that form it.

Retaining the labels of the non-interacting state means that the configurational entropy is unchanged in the interacting metal. (This also means that the quasiparticle distribution function is unchanged from the free particle result (see figure 3 (*a*)).) Each quasiparticle contributes additively to the total entropy of the system. This is not true for the energy. In an interacting system we must take into account that, unlike the free Fermi gas, the energy of individual excitations will not generally add to yield the total

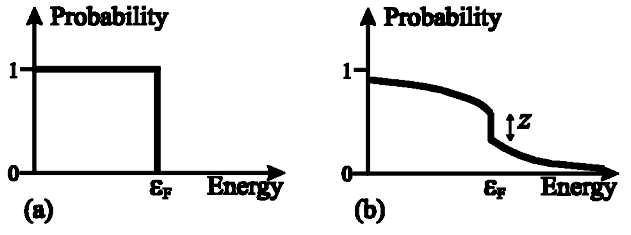


Figure 3. The probability that a state of a given energy is occupied at $T = 0$: (*a*) For electrons in a non-interacting system, or Landau quasiparticles in a Fermi liquid; (*b*) For electrons in an interacting Fermi liquid. Note the discontinuity at the Fermi energy ϵ_F remains, though reduced in size. The 'jump', z, is often considered as the order parameter of the Fermi liquid.

system energy. In Landau's theory, he accounted for the modified energy through two terms. First, when a quasiparticle moves there will now be a back-flow in the filled Fermi sea as the quasiparticle 'pushes' the ground state out of the way. This modifies the inertial mass of the quasiparticle $m \rightarrow m^*$. (Note that this is in addition to the effect of the crystal lattice—which produces a band mass which *can* be included in the free electron picture—and also the change induced by interactions with phonons.) Second, a quasiparticle's energy also depends on the distribution of other quasiparticles which Landau included via his 'f function'. The total energy of the interacting system is now expanded as a functional of the quasiparticle distribution $\delta n_{\mathbf{k}\sigma}$:

$$E = \sum_{\mathbf{k},\sigma} \frac{p_F}{m^*}(\hbar k - p_F)\delta n_{\mathbf{k}\sigma} + \frac{1}{2}\sum_{\mathbf{k}\mathbf{k}',\sigma\sigma'} f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}\delta n_{\mathbf{k}\sigma}\delta n_{\mathbf{k}'\sigma'} , \quad (1)$$

for an isotropic system. Using this one can then compute the equilibrium properties such as the specific heat and Pauli susceptibility we considered in the non-interacting problem above. One finds

$$c_v = \frac{1}{3}\frac{m^* p_F}{\hbar^3} k_B^2 T \quad (2)$$

$$\chi = \frac{m^* p_F}{\pi^2 \hbar}\frac{1}{1+F_0^a}\mu_B^2 . \quad (3)$$

These are similar to the free Fermi gas results except for the modified mass and the F_0^a term in χ which is related to the Landau f function and is known as a Landau parameter. Landau's theory also predicts new behaviour as the interaction between quasiparticles allows for collective modes of the system to develop. An example of these modes are the 'zero sound' oscillations of the Fermi surface whose restoring force is provided by the f function.

Before proceeding further we should check, as Landau did, that this procedure is internally consistent. Quasiparticles and holes are only approximate eigenstates of the system. In writing equation (1) we have neglected the possibility that measuring the energy with the Hamiltonian could change the quasiparticle distribution ($\delta n_{\mathbf{k},\sigma}$) itself. (That is to say that there remain matrix elements in the Hamiltonian which, when acting on a quasiparticle state, 'scatter' it into another state.) Recall that acting the Hamiltonian on a true eigenstate leaves the wave function unchanged up to a multiplicative constant (the eigenvalue). We can estimate the lifetime of these approximate eigenstates by considering the decay rate of a quasiparticle with energy ε above the Fermi surface at absolute zero. We can use Fermi's golden rule

$$\frac{1}{\tau_\epsilon} = \frac{2\pi}{\hbar}\sum_f |V_{if}|^2 \delta(\epsilon - \epsilon_F) , \quad (4)$$

where the sum is over the possible final states f. We will assume, for the time being, that the scattering matrix elements $|V_{if}|$ are constant and we will just enforce energy conservation and, crucially, the Pauli principle for quasiparticles. At absolute zero the only scattering allowed by the Pauli principle lowers the energy of the original quasiparticle by an amount ω by making an electron–hole pair in the filled Fermi sea. This process is illustrated in figure 4. The condition that the quasiparticle must scatter into an unoccupied state requires $\omega < \varepsilon$ In addition only occupied states within ω of the Fermi surface can absorb this energy by making a particle state above the Fermi surface. Thus our sum over final states becomes

$$\frac{1}{\tau_\epsilon} \sim \frac{2\pi}{\hbar}|V|^2 \int_0^\epsilon g_F \, d\omega \int_0^\omega g_F \, d\epsilon' \int_{-\infty}^{\infty} \delta(\epsilon - \omega - \epsilon' + \epsilon'') g_F \, d\epsilon'', \quad (5)$$

$$\sim \frac{\pi}{\hbar}|V|^2 g_F^3 \epsilon^2 , \quad (6)$$

where g_F is the density of states at the Fermi surface. Thus, close to the Fermi surface where ϵ is small, the quasiparticle is always well defined in the sense that its decay rate (ϵ^2) is much smaller than its excitation energy (ϵ). Far from the Fermi surface (i.e. large ϵ) adiabatic continuity will break down since the quasiparticle will decay before the interaction can be completely turned on. At temperatures above absolute zero the ambient temperature sets a minimum energy scale so for quasiparticles near the Fermi surface the scattering rate goes like T^2 (Abrahams 1954). Landau's picture of the interacting electron gas was therefore thought always to be valid provided one is concerned with small energy excitations and at low enough temperatures. This decay rate of quasiparticles is important in determining the transport properties of a Fermi liquid and results, for example, in a T^2 low temperature resistivity.

So far we have just discussed the properties of the Fermi liquid in terms of the quasiparticle. What of the electrons themselves? Adiabatic continuity tells us that in the quasiparticle wavefunction, there must remain a fraction of the original non-interacting excited state wavefunction

$$|\psi_{qp}(\mathbf{k}\sigma)\rangle = z^{1/2}|\phi_{el}(\mathbf{k}\sigma)\rangle + \text{particle} - \text{hole excitations etc} . \quad (7)$$

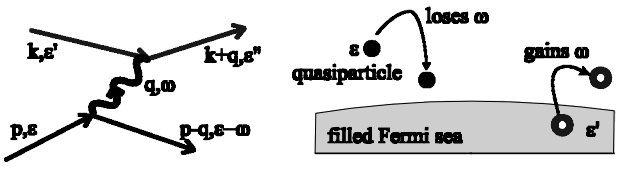


Figure 4. The scattering process for a quasiparticle with energy ε above the Fermi surface involves the creation of a particle–hole excitation.

That fraction, z, is known as the quasiparticle weight and, in a sense, plays the role of the order parameter of the zero temperature Fermi liquid state. A simple consequence of the step in the quasiparticle distribution at $T = 0$ is that, if one could analyse the electron distribution function, it too would show a discontinuous jump of size z at the Fermi momentum (see figure 3). A theoretical tool for following the fate of the original electrons in the interacting Fermi liquid is called the spectral function $A(\omega,\mathbf{k})$ (see, for example, Mahan 1990). It measures the probability that an electron with momentum $\mathbf{k}$ can be found with energy ω. In the non-interacting system single electrons are eigenstates of the system so the spectral function is a delta function $\delta(\omega - \epsilon_{\mathbf{k}})$. In the interacting system a given electron may take part in many eigenstates of the system and so the spectral function is spread out in energy. Nevertheless for momenta near k_F there is probability z that the electron may be found in the quasiparticle eigenstate with momentum k. So at $T = 0$ the electron spectral function in a Fermi liquid has a sharp peak at the new quasiparticle energy with width proportional to $(k - k_F)^2$, reflecting the finite lifetime, and an integrated weight under the peak of z (see figure 5).

Let us pause now to summarize the main features of Fermi-liquid theory. The success of the non-interacting picture of electrons is understood in terms of the existence of Fermi quasiparticles as approximate low energy eigenstates of the interacting system. We find that:

- Equilibrium properties have the free electron form but with modified parameters.
- The low energy eigenstates are fermion quasiparticles with a scattering rate $1/\tau \sim \max(\epsilon^2, T^2)$.
- z is the 'order parameter' of the Fermi liquid: the overlap of an electron and quasiparticle at the Fermi surface.
- New collective modes can also appear.

How well does this theory perform when tested against experiment?

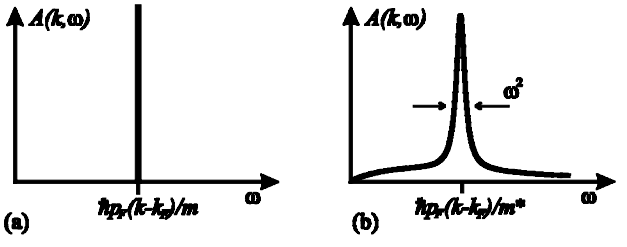


Figure 5. The spectral function: the probability that an electron with momentum k may be found with a given energy. (*a*) In a non-interacting system, electrons are eigenstates and so the probability is a delta function centred on the electron energy, $\epsilon(k)$. (*b*) In the Fermi liquid this probability is now spread out but retains a peak at the new quasiparticle energy. This peak sharpens as $k \to k_F$.

Thus far it looks as though a new free parameter has been introduced for each experiment. To test the theory we should demonstrate that experiments over-determine these free parameters. This is most straightforwardly done in ^{3}He which is isotropic. There it turns out that four experimental quantities (specific heat, compressibility, susceptibility and zero sound velocity) specify three of the Landau parameters and there is good internal agreement (see Wheatley 1970). In the metallic state the presence of a crystal lattice makes this test harder since the reduced symmetry allows many more Landau parameters. However, remarkably, recent experiments have confirmed the picture in one of the most strongly interacting metals known: UPt_3. The key feature of this material is the presence of uranium f electrons which are tightly bound to the atomic nucleus and are surrounded by a sea of conduction electrons. At high temperatures the f electrons behave as free magnetic moments with the classical Curie susceptibility (Frings *et al.* 1983). As the temperature is lowered the free spins start to become bound to conduction electrons to make up extremely heavy Landau quasiparticles (see figure 6). (The Kondo model (see Box 1) provides our theoretical picture for how this occurs—although in UPt_3 we of course have a dense periodic arrangement of magnetic ions rather than a single impurity.) The resulting Fermi liquid indeed recovers the free electron forms for the equilibrium

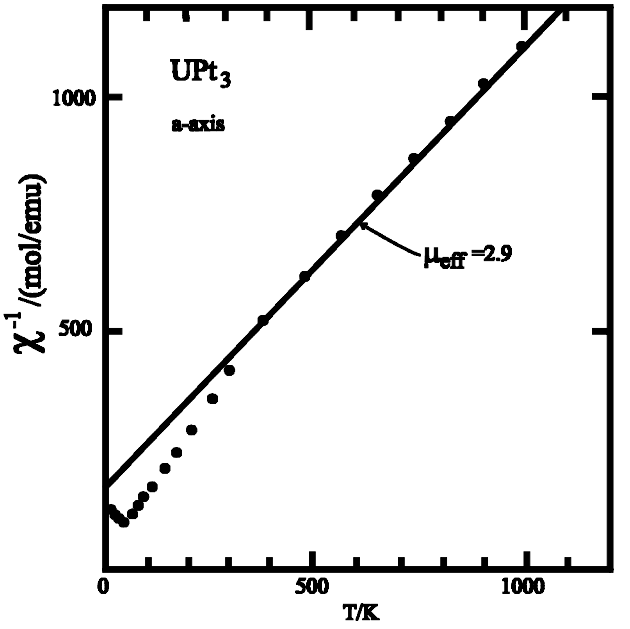


Figure 6. The inverse spin susceptibility of UPt_3 (after Frings *et al.* (1983)). At high temperatures we see the $1/T$ behaviour associated with free magnetic moments. This large moment becomes bound to the conduction electrons at low temperatures to form a heavy electron Fermi liquid.

Box 1. The Kondo Model

The Kondo model gives us a paradigm for understanding how a Fermi liquid arises in a number of the heavy-fermion metals. At its simplest, it describes the behaviour of a single spin-one-half ($S = 1/2$) magnetic ion in an otherwise non-interacting sea of electrons. The model is essentially zero dimensional since all the action occurs around the location of the ion. Typically this magnetic ion prefers to align its magnetic moment anti-parallel to that of any nearby electron (i.e. antiferromagnetically). Passing electrons scatter from the impurity and both can exchange their spin directions in the process. Kondo (1964) showed that, in contrast to most forms of scattering in a metal which normally reduce as the temperature is lowered, this spin-flip scattering grows logarithmically with decreasing temperature. Higher order perturbation treatments (Abrikosov 1965) predicted that the scattering would diverge at a finite 'Kondo' temperature (T_K)—an impossibility since the most scattering a single impurity can do is the unitarity limit when it behaves as an impenetrable sphere.

Understanding what really happens for $T < T_K$ required ideas of scaling (Anderson and Yuval 1971) and the renormalization group (Wilson 1975). It revealed that the proper way to think of Kondo's logarithm was to view the strength of the antiferromagnetic interaction between the ion and the electrons as effectively growing with decreasing temperature. At first this just enhances the scattering but, as the temperature is lowered further, the coupling becomes so strong the that magnetic ion prefers to bind tightly to a single electron and form an inert singlet state. The susceptibility associated with the impurity shows free-spin Curie behaviour at high temperatures but, as the singlet forms, the susceptibility saturates. The impurity specific heat peaks near T_K but then falls linearly to zero as the entropy associated with the impurity spin $[k_B \ln(2S+1)]$ becomes quenched in forming the $S = 0$ singlet. The scattering at $T = 0$ saturates at the unitarity limit and falls as T^2 for very small temperatures. The striking emergence of the Fermi liquid forms for these low temperature properties is due to the low energy excited states having a one-to-one correspondence with a weakly interacting Fermi gas (Nozières 1974). The Kondo temperature sets the effective Fermi energy of this local Fermi liquid.

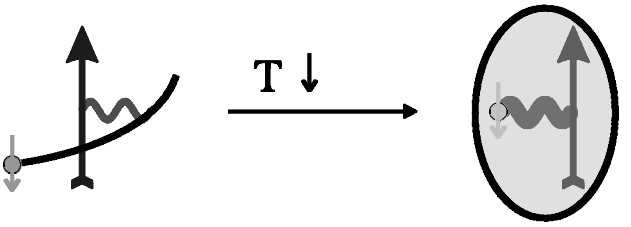


Figure B1. At high temperatures the Kondo impurity scatters conduction electrons but as the temperature is lowered the effective interaction between impurity and conduction electrons grows. Eventually a singlet bound-state is formed which acts as an inert potential scatterer.

The physics of the Kondo model exactly parallels asymptotic freedom and quark confinement in QCD. At high energies we see free spins (analogous to asymptotically free quarks with colour) but, as the energy is lowered, the spins become bound into singlets (analogous to the baryon colour singlets more familiar to us at terrestrial energies). The observation of Kondo type behaviour in UPt_3 and other heavy-fermion systems has been coined 'asymptotic freedom in a cryostat' (Coleman 1993).

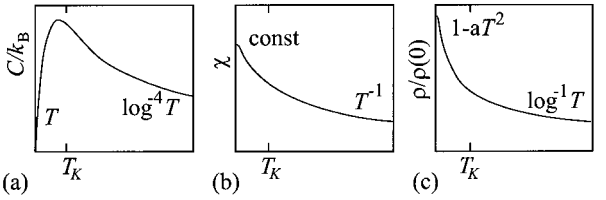


Figure B2. Impurity contributions to the (*a*) specific heat, (*b*) susceptibility and (*c*) resistivity in the Kondo model as a function of temperature. For $T \ll T_K$ these quantities recover the Fermi liquid forms as the impurity binds to the conduction electrons.

properties. The coefficient of the T-linear term in the specific heat (the intercept of the graph in figure 7 after Stewart *et al.* (1984)), though, is 450 mJ mol^{-1} K^{-2}—two orders of magnitude larger than that of a free electron gas. The effective mass of the quasiparticles has been vastly enhanced. This is adiabatic continuity pushed to the limits. To 'close' the theory one would like an independent check on the quasiparticle mass. This can be done from high magnetic field and low temperature measurements of the 'de Haas van Alphen effect'. At sufficiently high magnetic fields, quasiparticles can be driven around their Fermi surface by the Lorentz force. The quantization of these orbits leads to oscillations of the magnetization as a function of applied field: a kind of spectroscopy of the Fermi surface. Using this (Taillefer and Lonzarich 1988, Julian and McMullan 1998) one can map out the shape of the quasiparticle Fermi surface and, from the temperature dependence, deduce the

quasiparticle effective mass. These masses and Fermi surfaces are shown in figure 7. Two key observations emerge even in this most interacting of environments:

(1) The volume of the Fermi surface includes the *f* electrons.
(2) The measured quasiparticle mass accounts for the enhanced specific heat.

Both these observations confirm the success of Fermi-liquid theory.

3. The mystery of the cuprate superconductors

If Fermi-liquid theory gives such a good account of the metallic state, why look for alternatives? The reason is the discovery of metals where the fermion quasiparticle does not seem to reflect the character of the measured low energy eigenstates. In this sense, the electron (or electron-like quasiparticle) may no longer be the appropriate way to think of the low-lying excitations. The prime example is the metallic state of the copper oxide superconductors and so it seems fitting to motivate the search for alternative theories of metals by summarizing the puzzles presented by these materials.

The superconducting cuprates encompass almost thirty distinct crystalline structures (Shaked *et al.* 1994) and contain upwards of three different elements. They are united by the common feature of a layer structure of CuO_2 planes. In their pristine state these compounds are typically not metals—they are insulators with antiferromagnetic order. (The magnetic moment on the copper site alternates in direction as you move from one site to its neighbour.) This in itself is a signature that the strength of the interaction between electrons is important since simple electron counting in the absence of interactions would suggest that these materials should have a half filled band and hence metallic properties (see Box 2). In order to make these materials metallic (and superconducting at low temperatures) one removes some electrons from each copper-oxide layer by doping with another element which typically resides between the copper oxide planes. The antiferromagnetism then disappears and the material becomes a metallic conductor.

The metallic behaviour that arises is characterized by significant anisotropy. The electrical resistance perpendicular to the CuO_2 plane direction can be up to one thousand times greater than that for currents carried in the planes (Ito *et al.* 1991, Hussey 1998). Many people have stressed the importance of the effectively two-dimensional nature of the metallic cuprates and this is in part what has prompted the search for unusual metals in low dimensions. As I have already hinted, the metallic state itself is unusual. Space precludes a detailed analysis

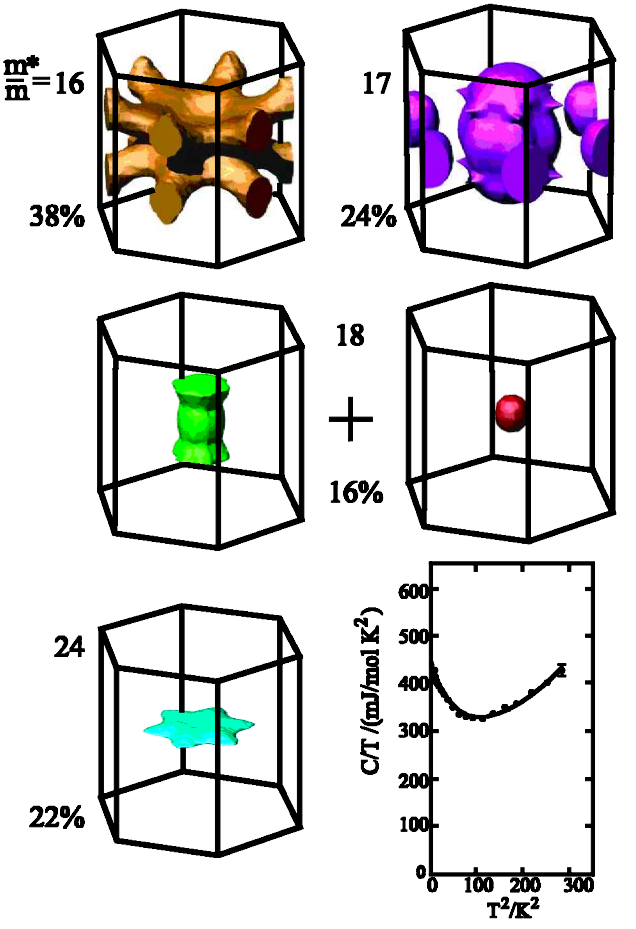


Figure 7. The consistency of the Fermi liquid description has been demonstrated in UPt_3. The five Fermi surface sheets (from Julian and McMullan 1998) and effective mass of the quasiparticles have been mapped out by de Haas van Alphen measurements (Taillefer and Lonzarich 1988). They confirm that the $5f^3$ electrons are absorbed into the Fermi liquid and that the quasiparticle masses are consistent with the mass enhancements measured in specific heat (after Stewart *et al.* (1984)). The percentages reflect the contribution from quasiparticles on each sheet to the total specific heat.

of all of the anomalous properties so I will concentrate on just a few observations.

The first noticed peculiarity is the extraordinary linear temperature dependence of the resistivity (Gurvitch and Fiory 1987) at dopings which maximize the superconducting transition temperature (illustrated in figure 8). Optical measurements confirm that this is due to a scattering rate which is proportional to temperature (Forro *et al.* 1990). While this is clearly different from the T^2 scattering that Landau's theory might predict, we should remember that the usual quasiparticle scattering only becomes apparent at very low temperatures which, in this case, is obscured from view by the transition into

Box 2. The *tJ* model

In the *tJ* model we imagine a highly simplified view of a CuO_2 plane. It starts from a picture of the parent insulating compounds with a square lattice of single atomic orbitals each with exactly one electron. The system is said to be half-filled since the Pauli principle would allow a maximum of two electrons in an orbital. Normally at half-filling we would expect a metallic state since any electron can move through the system to carry current by hopping onto a neighbouring site. Now we imagine turning on the inter-electron repulsion so that it becomes energetically unfavourable for more than a *single* electron to occupy a given atomic site. Now the half-filled case is an insulator because any electron moving to a neighbouring site already finds an electron there and pays the price of the repulsive interaction. There is an energy gap to make a current carrying state and we have a 'Mott insulator'. This explains the insulating nature of the parent cuprates. In fact no electron really likes to be fixed on a single site—it is like being held in a small box and its kinetic energy is high. This can be lowered if the electron made 'virtual' tunnelling hops onto the neighbouring occupied sites and back. This the electron can only do if the neighbouring site has the opposite spin (the Pauli principle remains absolute). So we see that the interactions also favour the antiferromagnetic arrangement of spins seen in the parent compounds.

The *tJ* model describes what happens on doping. We now remove some electrons so that there are empty sites in the lattice.

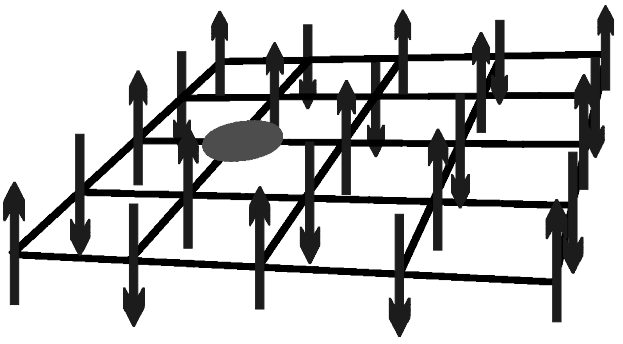


Figure B3. The *tJ* model describes the competition between hole motion and antiferromagnetism in a doped Mott insulator.

The system becomes a metal since electrons near empty sites can move without restraint (and lower their kinetic energy by an amount t. The neighbouring electrons would still like to remain antiferromagnetically aligned with strength J and at no time must any site contain more than one electron. The presence of this constraint means that there is no small parameter in the theory with which one can perform perturbation theory. This is the basic physics of the *tJ* model and, because the moving electrons disorder the magnetism, we see that the t and the J represent competing interactions. While there are many fascinating proposals for understanding the physics of this model, there are few definitive results.

the superconducting state. Electron–phonon scattering usually gives a linear resistivity from moderate temperatures upwards (which is why we often use the resistance of a platinum wire as thermometer). However in the case of the cuprates it seems that this scattering is purely electronic in origin since microwave measurements show the scattering rate plummeting on entering the superconducting state (Bonn *et al.* 1993). So the measured resistivity presents us with two puzzles: what causes the linear T scattering and how to explain the absence of the phonon scattering?

We have discussed the scattering of electric currents but one can also measure the scattering of currents generated in a Hall effect experiment. Typically both electric currents and Hall currents should measure the same scattering rate. In figure 9 we see that the scattering rate from Hall currents rises quadratically with the temperature in absolute contradiction to the resistivity experiments (Chien *et al.* 1991). How can a single quasiparticle have two relaxation rates?

Furthermore there is the puzzle of how many charge carriers there are in these metals. Some experiments, like penetration depth in the superconducting state (Uemura *et al.* 1991), suggest a number proportional to the (small) number of holes made by doping the insulating state. Yet other experiments such as angle-resolved photo-emission see a Fermi surface containing the large total number of conduction electrons in the system (Campuzano *et al.* 1990).

The proposed solutions to these questions remain highly controversial and have led to some very exciting and far reaching ideas which, even if they do not ultimately find fulfillment in the physics of the cuprates, will certainly resurface in the physics of other compounds. What makes the subject exciting and at the same time difficult is the absence so far of any solvable model describing how interactions give rise to the metallic state of the cuprates. The simplest model we have is the so-called *tJ* model (see Box 2) for which no solution exists outside one dimension (see later).

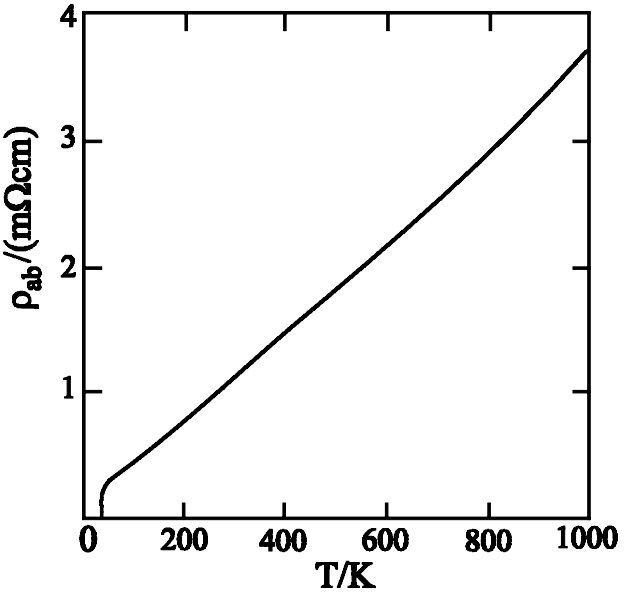


Figure 8. The resistivity of $La_{1.85}Sr_{0.15}CuO_4$ is linear in the temperature (after Takagi *et al.* (1992)) and indicates a scattering rate for electrical currents proportional to temperature.

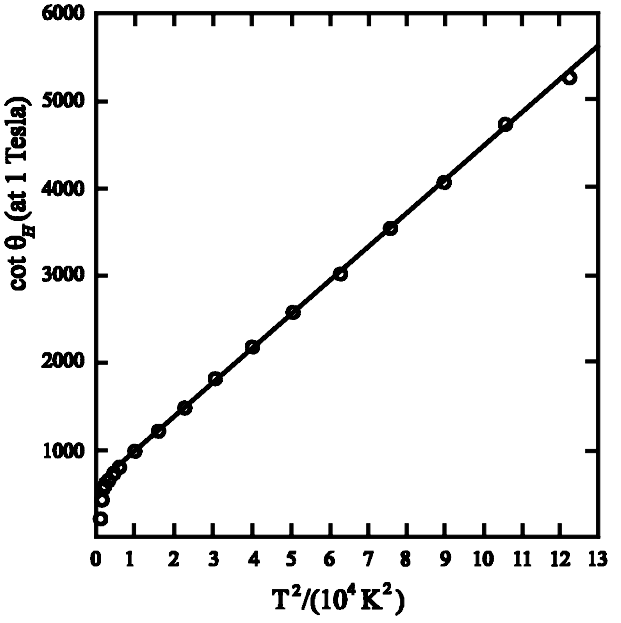


Figure 9. The inverse Hall angle in $La_{1.85}Sr_{0.15}CuO_4$ in (after Harris *et al.* (1995)). This measures the scattering rate for Hall currents and shows it to be proportional to T^2. This is in marked contrast to the scattering of electric currents which is proportional to T. The puzzle is how can a single quasiparticle have two distinct scattering rates?

One might be surprised that I have underplayed the relatively large values of the superconducting transition temperature for which the high temperature superconductors received that epithet. This is not because the superconductivity is not important or unusual. (These are the first materials where the superconducting order parameter has been established to have a symmetry different from the usual *s* wave form (see Annett *et al.* 1996)). It is because the understanding of superconductivity usually requires an understanding of the metallic state from which it forms. We should note that our understanding of conventional superconductivity relies on a Fermi-liquid starting point. The pairing instability is a consequence of the sharp discontinuity in occupation at the Fermi surface which the fermion quasiparticle picture provides (Bardeen *et al.* 1957). The cuprates are not alone in exhibiting superconductivity from an unusual metallic state. There is another uranium alloy, UBe_{13}, which shows superconductivity (Ott *et al.* 1983, Bucher *et al.* 1973) but with little evidence of a well formed metallic Fermi-liquid state. It is clear that the properties of new superconductors and their unusual metallic states are intimately linked.

If the task is to find new descriptions of the metallic state, where should we begin? There have been many speculative suggestions, but in this article I want to focus on examples of non-Fermi liquids which we believe we do understand at least from a theoretical view point. In fact all four of the examples I will discuss have been applied (loosely in some cases) to account for the physics of the cuprates or uranium alloys. These examples show a great richness in behaviour although they may appear at first sight to be rather artificial 'toy' models. This is a consequence of our lack of the mathematical tools with which to treat problems where the interaction between electrons is strong and the tools that we do have are often only applicable to systems in reduced dimensions. However, the ingenuity of material scientists means that these toy systems can often be realized in nature by the clever tailoring of materials. Whereas in the past the theoretical physicist's job has often involved finding the simple model that best describes the physics of a complex material, now we have the exciting possibility of the material scientist developing materials to demonstrate the theorist's model.

Since we are here concerned with the failure of Fermi-liquid theory, it is as well to mention one clear example where adiabatic continuity breaks down. Once level crossings occur in the spectrum then it is no longer possible to follow the labels through from the non-interacting case. Typically this happens when there is a phase transition in the system such as the formation of a superconducting state when electron bound states are favoured. In this review we will be primarily interested in how the Fermi liquid can fail within the normal metallic state so we will not be considering such phase transitions.

However, our first example of a non-Fermi liquid will be one where the *approach* to a phase transition can disrupt the Fermi-liquid state by destroying the Landau quasiparticle.

4. Metals near a quantum critical point: destroying the Landau quasiparticle

In our discussion of Landau's approach we showed that a quasiparticle close to the Fermi surface was a long lived eigenstate by determining the decay rate. In doing this we assumed that the matrix elements for scattering were independent of the momentum and energy transfered. The Pauli principle confines all scattering particles to the vicinity of the Fermi surface so there is little scope for large transfers of energy in scattering. In the limit of small energy transfer our assumption is often valid. However, when a system approaches a second order phase transition we know that fluctuations of the order parameter slow down and occur over increasingly long wavelengths. A moving quasiparticle can then easily generate a large disturbance in the medium which can, in turn, affect other quasiparticles in the vicinity dramatically enhancing the scattering cross-section. This effect is limited by the ordering temperature which locks the fluctuations into a long range ordered state. Below this temperature again our initial assumptions remain valid and the Landau quasiparticle is saved. Recently much work (both theoretical and, as we will see, experimental) has explored phase transitions occurring at $T = 0$ K (Hertz 1976, Moriya 1985, Millis 1993). While the types of phase transition will be familiar (for example from a paramagnetic to a magnetic state) they are unusual in that Nernst's theorem tells us that the entropy should always be zero at zero temperature. A zero temperature phase transition must therefore be a transition between two ordered states. For our purposes the most important feature will be that the quasiparticle scattering cross-section can now grow without limit ultimately destroying the consistency of Landau's Fermi liquid picture.

A zero temperature phase transition occurs at a quantum critical point—so called because quantum mechanics determines the fluctuations of the order parameter. It turns out that some of the physical properties near a temperature phase transition can be determined simply by using Fermi's golden rule, together with the appropriate matrix elements. The matrix element encapsulates the scattering mechanism (in this case the long range fluctuations of the order parameter which is trying to develop at the phase transition). In fact I use the example of the magnetic interaction between moving charges because the matrix elements can be computed from Maxwell's equations. (It turns out that the matrix elements are the same as those near a zero temperature transition to a ferromagnetic state.) The following section is meant for those readers who would like to see where some of the non-Fermi liquid properties come from. However, it may be omitted if you just want to know the final results.

4.1. *Properties near a quantum critical point*

To see in some detail how Landau's original argument is spoiled near a quantum critical point we must revisit our Fermi golden rule expression for the decay rate. I will now do a change of variable and express the same quantity as equation (6) in terms of the momentum and energy transfered in any scattering process. The result (valid in dimension d) is

$$\frac{1}{\tau_\epsilon} = \frac{2\pi}{\hbar}\int_0^\epsilon g_F \omega \, d\omega \int_{\omega/\hbar v_F}^{2k_F} \frac{q^{d-1}\, dq}{(2\pi/L)^d} \frac{|D(q,\omega)|^2}{(\hbar v_F q)^2} \,. \qquad (8)$$

The integral over ω is simply the number of possible hole excitations that can be created. The appearance of ω in the lower limit of the momentum integral, q, appears because a minimum momentum must be transfered to give a change in energy of ω. The integration over the direction of the momentum has already been performed and gives the factor of $(\hbar v_F q)^2$ in the denominator reflecting the increased time available for small deflections. Finally $D(q,\omega)$ is the matrix element for the scattering process. If this is independent of q and ω then the q integral is not sensitive to the value of the lower limit and is independent of ω. The subsequent integration over ω recovers the usual $\tau_\epsilon^{-1} \sim g_F \epsilon^2$ result we had before.

Already at this point we can see that something interesting happens in one dimension ($d = 1$). The singular nature of the q integral, even when the matrix element $D(q,\omega)$ is constant, leads to a decay proportional to ϵ to this order. This is a signal of the breakdown in adiabatic continuity since there is no limit when the quasiparticle energy is well defined: there are *no* one-dimensional Fermi liquids! We will discuss the nature of metals that do arise in $d = 1$ later. In higher dimensions we need to make $D(q,\omega)$ more singular at low q in order to destabilize the Fermi liquid by this route.

Singular interactions are a consequence of long range force laws (large distances correspond to small wavenumbers in reciprocal space). The usual Coulomb force itself adds to the Hamiltonian a term:

$$\frac{\rho_e(\mathbf{r})\rho_e(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \longrightarrow D_c(q) = \frac{\rho_e(q)\rho_e(-q)}{4\pi\epsilon_0 q^2} \,, \qquad (9)$$

where ρ_e is the density of electronic charge. This has exactly the sort of singularity at small q one might expect to destabilize the Fermi liquid. In fact the Fermi liquid remains stable because the collective behaviour of the other electrons screens the long range Coulomb repulsion. Any local build up of charge causes the nearby electrons to move away revealing more of the background lattice of

positive ions. This neutralizes the charge build up beyond the Debye–Hückel screening length $\xi \sim (\epsilon_0/2\pi e^2 g_F)^{1/2}$ (Debye and Hückel 1923). Thus our once long-ranged interaction is actually a short distance Yukawa-type potential and is perfectly innocuous as far as the quasiparticle is concerned. The screened Coulomb interaction no longer diverges as $q \to 0$:

$$\frac{\rho_e(\mathbf{r})\rho_e(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} e^{-|\mathbf{r}-\mathbf{r}'|/\xi} \longrightarrow D_{sc}(q) = \frac{\rho_e(q)\rho_e(-q)}{4\pi\epsilon_0(q^2+\xi^{-2})} . \quad (10)$$

(Strictly one needs to consider the frequency dependence of the electron's response (i.e. dynamical screening) to determine the influence of the Coulomb interaction on the Fermi liquid (Silin 1957, Pines and Nozíeres 1966).)

It turns out, however, that the Amperean interaction between moving charges (more familiar to us as the force between two current carrying wires) is much more dangerous to the Fermi liquid (Holstein *et al.* 1973, Reizer 1989, 1991). These forces turn out to be much weaker than the Coulomb law and their danger is only apparent at extremely low energies and hence at inaccessibly low temperatures. However an almost identical form of matrix element arises near a ferromagnetic quantum critical point and this *is* experimentally realizable. Since deriving the matrix element at the quantum critical point is beyond the scope of this article, I will use the Amperean interaction as my example of a singular interaction. In this case seeing why the force law is singular is a straightforward application of Maxwell's equations.

The interaction between moving charges is due to the local magnetization which they set up. In the case of the Coulomb interaction the potential energy term in the Hamiltonian is just $\phi(\mathbf{r})\rho_e(\mathbf{r})_e$: the charge density times the electrostatic potential. The term we need for the Amperean force law is the product of the local current density and the *vector* potential: $\mathbf{A}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r})$. For the Coulomb case we know that a local charge produces a $1/r$ potential. To find the vector potential from a local current density, $\mathbf{j}$, we consider the fields generated by its magnetization $\mathbf{m}$ in the presence of the currents from other electrons in the metal $\mathbf{J}$. Using Maxwell's equations

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} , \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mu_0\nabla\times\mathbf{m} , \quad (11)$$

and the definition of the conductivity in the metal $\mathbf{J} = \sigma\mathbf{E}$, we can find the magnetic field $\mathbf{B}$ and hence the vector potential $\mathbf{B} = \nabla\times\mathbf{A}$. It is usual to Fourier transform the result to obtain the appropriate interaction term

$$D_{Amp}(q,\omega) = \mathrm{A(j)}\cdot\mathrm{J} = \frac{\mu_0 \mathrm{j}_{q,\omega}\cdot\mathrm{j}_{-q,-\omega}}{q^2 + \mathrm{i}\omega\sigma\mu_0 - \omega^2/c^2} . \quad (12)$$

(The ω^2/c^2 term does not play an important role here and so we will drop it from now on.) By comparing the equation for this interaction with that of the screened Coulomb interaction (equation (10)) we see that the appropriate 'screening length' for the current–current interaction is the skin depth

$$\xi \sim 1/(\mathrm{i}\omega\sigma\mu_0)^{1/2} . \quad (13)$$

Unlike the Coulomb case considered previously, this screening length *diverges* at low ω (i.e. low energies) and so fails to usefully suppress the growing scattering matrix element at low energies. For the Amperean interaction, the quasiparticle scattering cross-section grows without the limit and this destroys the Fermi liquid. Since this type of interaction is always present in a metal, why is Fermi-liquid theory a good description of any metal? The answer lies in the overall scale of the interaction. Comparing the ratio of the Coulomb and Amperean interactions we see that (for currents carried by quasiparticles near the Fermi surface moving with velocity v_F)

$$\frac{D_{Amp}}{D_{sc}} \sim 4\pi\epsilon_0\mu_0\frac{j_e^2}{\rho_e^2} \sim 4\pi\left(\frac{v_F}{c}\right)^2 . \quad (14)$$

Thus the current–current interaction is 10^6 times weaker than the Coulomb interaction which is why its effects are only likely to be visible at micro Kelvin temperatures.

We can see what behaviour might be produced, though, by computing the quasiparticle lifetime from equation (8) with D_{j-j} as found above. In a clean metal with no impurities the conductivity is limited, not by the mean-free-path, but by the wavevector: $\sigma \sim 1/q$. The skin depth enters the so-called anomalous regime and behaves like $(\omega/q)^{-1/2}$ and this is the case we will consider. Doing the integral of equation (8) may look tricky but the singular nature makes everything simple. We are only trying to extract the energy dependence of the decay rate so we will neglect the prefactors. We note that as q decreases then D_{Amp} grows as $1/q^2$ until q becomes smaller than the inverse skin depth when D_{Amp} saturates. This happens when $q^2 \sim \omega/q$ (i.e. when $q \sim \omega^{1/3}$). Since $\omega^{1/3}$ is always greater than the lower limit of the integral (ω) as $\omega \to 0$ we can use it as the lower limit. We can then find the energy dependence of the scattering rate by considering

$$\frac{1}{\tau_\epsilon} \sim \int_0^\epsilon \omega\,\mathrm{d}\omega \int_{\omega^{1/3}} \frac{q^{d-1}\,\mathrm{d}q}{q^2}\left(\frac{1}{q^2}\right)^2 , \quad (15)$$

$$\sim \epsilon \sim T , \qquad \text{for } d = 3 \text{ at temperature } T . \quad (16)$$

There is now no regime where the Landau quasiparticle is sufficiently long-lived to count as an approximate eigenstate. In the language of adiabatic continuity, switching on the interaction adiabatically takes longer than the lifetime of the eigenstate itself and so one can never continue from the non-interacting state to the interacting one. When the scattering rate goes linearly to zero with the energy as in this case, we have a 'marginal Fermi liquid' (Varma *et al.* 1989).

Surprisingly, despite the absence of quasiparticles, we can still use this calculation to determine some properties of this non-Fermi liquid metal: the temperature dependence of the resistivities and heat capacity. The decay rate (which is related to the resistivity) and the effective mass (which gives the heat capacity) are inextricably linked in this example. This is because the scattering of quasiparticles can not help but produce some back-flow which contributes to the effective mass and the quasiparticle energy. The decay rate may be viewed as an imaginary component of the energy in the time evolution of the wavefunction $\sim\exp(-i\epsilon t/\hbar)$. When the decay rate is non-analytic as $\epsilon\rightarrow 0$ then one can obtain the real part from the imaginary part through the Kramers–Kronig relation. The essence of this is that if the quasiparticle scattering rate is T^{α} (with α some fractional power), the heat capacity also has a T^{α} dependence. In the case we have just considered $\alpha = 1$. This is non-analytic at the origin since, as a decay rate is always positive, it should really be $\tau_{\epsilon}^{-1} \sim |\epsilon|$. In that case the heat capacity becomes $T \ln T$. So we have a logarithmic correction to the usual linear in T specific heat capacity.

The other quantity which we can compute is the resistivity. One might be tempted to conclude that, since the scattering rate is linear in temperature, then the resistivity should also be proportional to temperature. However this fails to account for the effectiveness of the scattering at destroying electrical current. Small q scattering may destroy the quasiparticle but it is not effective at removing momentum from the net flow of current. For that to happen large angle scattering must occur. The transport lifetime takes this into account and is obtained from the same expression as equation (8) with an additional factor of $(1-\cos\theta)\sim q^2/k_{\mathrm{F}}^2$ where θ is the scattering angle. Doing this above gives a resistivity of $T^{5/3}$—a stronger temperature dependence than the usual T^2 in a Fermi liquid. The other quantity we have discussed, the Pauli susceptibility, can not be obtained from our Fermi golden rule expression and requires more analysis to obtain.

4.2. *Examples of quantum critical points*

Summarizing these results, we have shown how the interactions between currents will ultimately destroy the Fermi-liquid state in any metal at very low temperatures. The new behaviour that we expect to see includes

- a marginal quasiparticle scattering rate: $\tau_{\epsilon}^{-1} \sim \epsilon \sim T$,
- $C_V \sim T \ln T$,
- $\chi \sim \ln T$ (not proved here),
- $\rho(T) \sim T^{5/3}$.

It is a curious paradox that, while Landau's arguments would suggest that the Fermi-liquid description is valid in the low temperature limit, interactions like this actually provide a low temperature bound on its stability. The Kohn–Luttinger instability (Kohn and Luttinger 1965) to a superconducting state similarly acts in a general way to prevent one ever obtaining a $T = 0$ Fermi liquid. In most metals these effects are unobservable, but there are a growing number of cases where new types of singular interactions can lead to a non-Fermi liquid state which *is* observed. I have already indicated that metals near a quantum critical point provide us with such examples and so too do electrons in a half-filled Landau level and I will now discuss what can be seen in experiment.

When a metal is on the verge of a ferromagnetic instability then one finds that the effective interactions between quasiparticles have exactly the same form as they do in the Amperean case we have just considered. One example of this is in the compound MnSi (Pfleiderer *et al.* 1997). It becomes an almost ferromagnetic helical magnet at 30 K but under pressure this Curie temperature is lowered, until at 14 800 atmospheres the magnetism has been completely 'squeezed' out of the system. It has not yet been possible to measure the heat capacity under such extreme conditions but resistivity measurements can be done. In figure 10 we see how the resistivity diverges at low temperatures when compared to the expected Fermi-liquid form until the phase transition to the ferromagnet occurs. When this phase transition is pushed to absolute zero we have a true quantum critical point and this divergence proceeds without limit. The resistivity takes on a $T1.6 \pm 0.1$ dependence similar to the new $T^{5/3}$ form we proved earlier. Similar behaviour has also been seen in high pressure experiments on $ZrZn_2$—another ferromagnet with a low Curie temperature (Grosche *et al.* 1995).

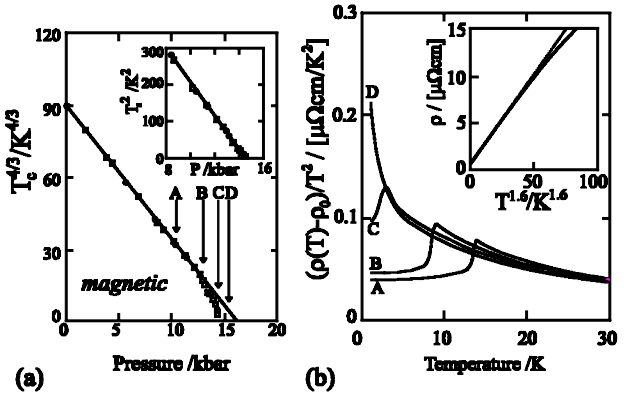


Figure 10. MnSi: a low-temperature long-wavelength helical magnet. (*a*) The phase diagram showing that the quantum critical point occurs at 14.8 kbar. (*b*) As the quantum critical point as approached, the resistivity takes on a $T0.6 \pm 0.1$ temperature dependence showing that the quasiparticles are more strongly scattered than in a normal Fermi liquid (after Pfleiderer *et al.* (1997)).

There are a number of other known non-Fermi liquids which arise from singular interactions. A very similar form of interaction occurs when a two-dimensional electron gas is subjected to high magnetic fields, but it comes from quite a different source. This perhaps is the most unusual Fermi liquid we know, although again the singular interactions imply that it is not truly a Fermi liquid at the lowest temperatures.

At high magnetic fields we see the fractional quantum Hall effect (Tsui *et al.* 1982). Electrons in a magnetic field and confined to two dimensions develop a discrete quantized energy spectrum where each level can hold a macroscopic number of particles—a number which depends on the strength of the magnetic field. When the lowest energy level is partially filled by a fraction with an odd denominator (like 1/3 full), the ground state shows unusual stability. The Hall effect develops a plateau and becomes independent of a field for a small range of nearby fields as the electron fluid is reluctant to move away from this stable point. The excitations of this insulating state carry fractions of the electronic charge (Laughlin 1983) and are a fascinating area of active research which I will not detail here as I am concentrating on metallic states. The ground state can be thought of as being formed from bound states of electrons with quanta of flux (Jain 1989, 1992) making 'composite fermions'. Whatever magnetic field is left unbound after the composite fermions form, is the effective field experienced by the composite fermions. They then undergo a conventional integer quantum Hall effect. Between the quantum Hall plateaux the electron gas passes through a metallic phase. When the lowest energy level is exactly half filled the composite fermions try to form a metallic state where there is no effective field remaining. We then have a Fermi liquid of composite fermions (Halperin *et al.* 1993). The residual interactions in this metal can also have a singular form and so affect its properties. These come from an interaction between currents and charge density which is left over from approximations of binding magnetic flux to the underlying electrons. This type of interaction (coupling a vector to a scalar) is not usually allowed in a metal but occurs here in the presence of a magnetic field. It is the absence of a long range interaction (which would otherwise suppress the fluctuations of density) that can cause the effective coupling to be singular. Nevertheless one sees strong evidence for a well formed Fermi liquid in the experiments (Willett 1997).

Perhaps the most puzzling of the systems with a singular interaction are antiferromagnetic quantum critical points. There exist a growing family of metals with very low Néel temperatures below which antiferromagnetic order develops. An example is $CePd_2Si_2$ where the Néel temperature can be squeezed to zero in pressures of 28 000 atmospheres (see figure 11). Similar arguments to those presented above can be used to compute the expected temperature dependence of the resistivity: it turns out to be $T^{3/2}$. However, unlike the ferromagnetic case, the power law that is observed is $T^{1.2}$ (Grosche *et al.* 1996) (see figure 12). Why this should be is presently not understood but is potentially a question of fundamental importance. The cuprate superconductors are also systems close to antiferromagnetism as we have seen, and it has been argued that these two puzzling phenomenon are linked. Tantalizingly, this system and others close to antiferromagnetism (Mathur *et al.* 1998) also show superconductivity at the quantum critical point lending weight to a connection between this type of

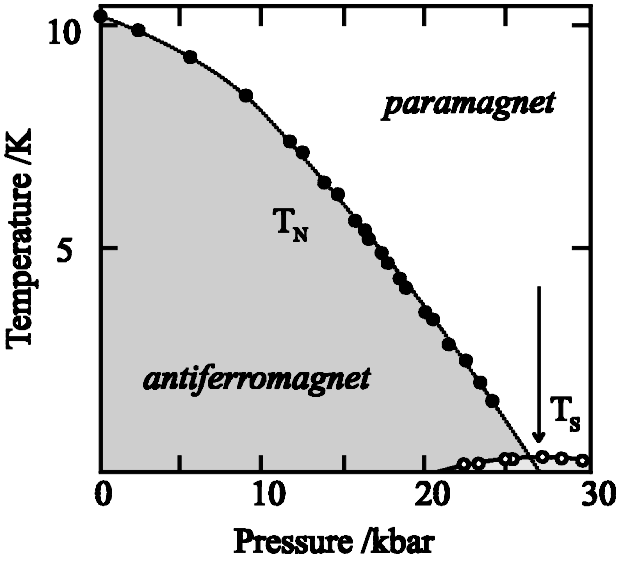


Figure 11. $CePd_2Si_2$: a low temperature antiferromagnet. Under pressure the antiferromagnetism can be suppressed to zero temperature giving a quantum critical point. Not only do we see non-Fermi liquid behaviour here but also there is a superconducting transition (after Julian *et al.* (1996) and Mathur *et al.* (1998)).

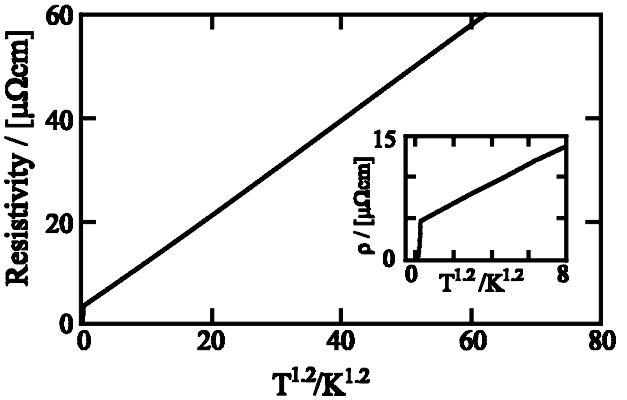


Figure 12. The resistivity of $CePd_2Si_2$ at the critical pressure (28 kbar). The observed temperature dependence, $T^{1.2}$, is seen over two decades of temperature. It has not yet been explained. (Data after Grosche *et al.* (1996).)

quantum critical point and the cuprates. However the superconducting transition is at 0.4 K which leaves this scenario needing to explain why T_c is so high in the cuprates.

In our discussion thus far we have used only the resistivity as the signature of non-Fermi liquid physics. Primarily, this is because all of the above materials require pressures so high to reach the critical point that thermodynamic measurements (e.g. specific heat or Pauli susceptibility) are rather difficult. This can be overcome by using doping rather than pressure to tune the metal to a quantum critical point. One such example is $CeCu_{6-x}Au_x$ (see von Löhneysen 1996a, b) which is paramagnetic with $x = 0$ but on adding a small mount ($x = 0.1$) of gold, the metal develops antiferromagnetism (see figure 13). Here one can look at all the familiar indicators of Fermi-liquid behaviour and show how they deviate near the critical point. The specific heat shows a $T \ln T$ relation (see figure 14), the resistivity is linear in temperature and the Pauli susceptibility diverges as $-\ln T$ at low temperatures. The danger with doping as a tuning mechanism is that the metals are now off stoichiometry and, as such, one must wonder about the role of disorder near the critical point. This is theoretically an open problem. Nevertheless, the experimental situation is clear: we have a well tried route to the non-Fermi liquid at a quantum critical point.

If all that was known about non-Fermi liquids was how the quasiparticle could be destroyed by singular interactions, we would seem to have found only the exception to prove the general rule. However in one dimension we see a radical new type of metallic behaviour where completely new types of particle emerge to replace Landau's Fermi quasiparticle.

5. Luttinger liquid: the Bose quasiparticle

We have already seen that our general scattering rate argument would predict an absence of Fermi liquids in one dimension even with a constant matrix element. Considering higher order terms only makes matters worse. All is not lost however, for a new type of adiabatic continuity has been proposed by Haldane (1981) which gives us the possibility of quantifying the new metallic state that emerges in its place: the Luttinger liquid. Discussing one dimension may seem rather esoteric when we live in a three-dimensional world. In fact many systems, from the 'blue bronze' molybdenum alloys to some organic Bechgaard salts have properties which are highly anisotropic. Electron motion is essentially confined to one dimension by the very low probability of the electron hopping in the two remaining directions. It is in this type of system that we have the possibility of seeing a Luttinger liquid state develop.

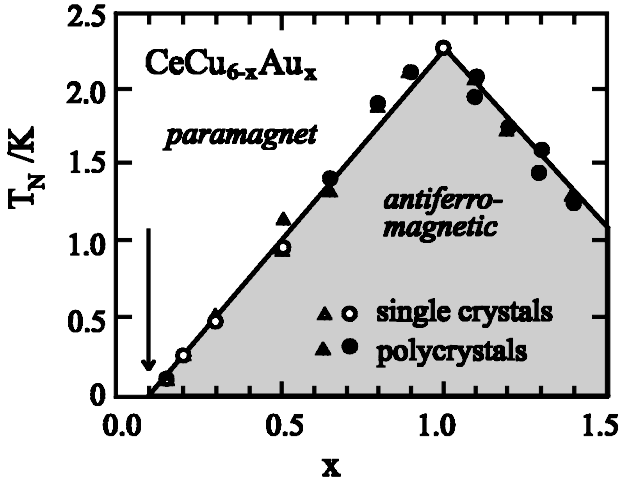


Figure 13. The $CeCu_{6-x}Au_x$ system shows an antiferromagnetic quantum critical point driven by gold doping. Using doping to tune to the critical point opens up the possibility of doing more measurements on the new metallic state. It comes at the price, though, of increasing the sample disorder which can complicate the theoretical understanding of the non-Fermi liquid state (after Pietrus *et al.* (1995)).

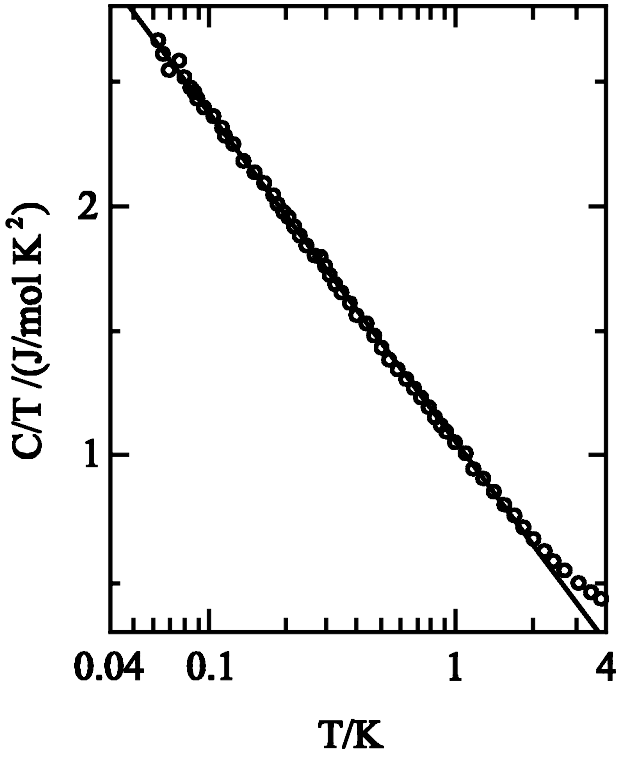
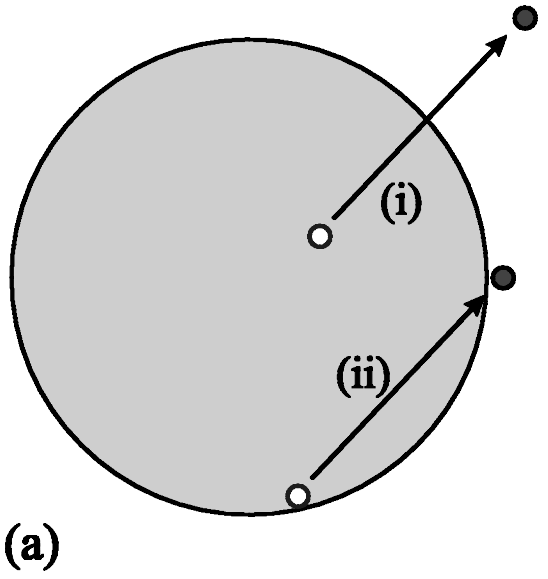


Figure 14. The specific heat of $CeCu_{5.9}Au_{0.1}$ at the quantum critical point. The heat capacity shows a $T \ln T$ form indicating that the thermodynamics of the non-Fermi liquid are totally changed by the proximity to the critical point. (Data after von Ľohneysen *et al.* (1996a, b).)



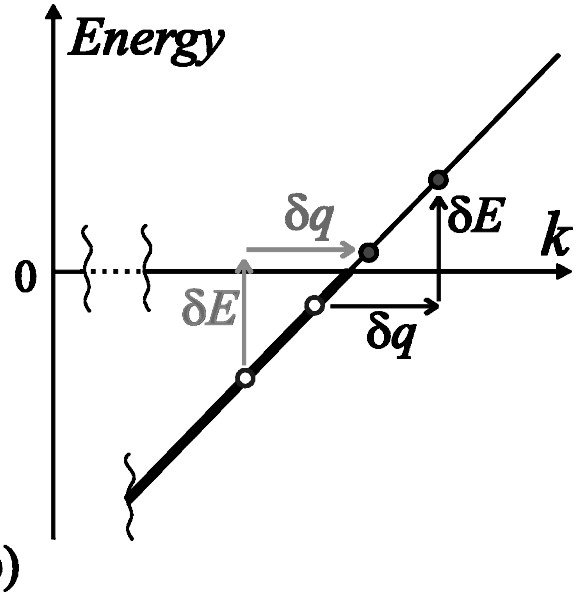


Figure 15. One dimension has the special feature that all particle–hole excitations with a given momentum q have the same energy. (*a*) In high dimensions one can make (i) a high energy excitation or (ii) a low energy one depending on the whether q is normal or transverse to the local Fermi surface. (*b*) In $d = 1$ there is only one direction in the problem and so fixing δq determines the energy change δE. This leads to density waves being the proper description of physics in one dimension.

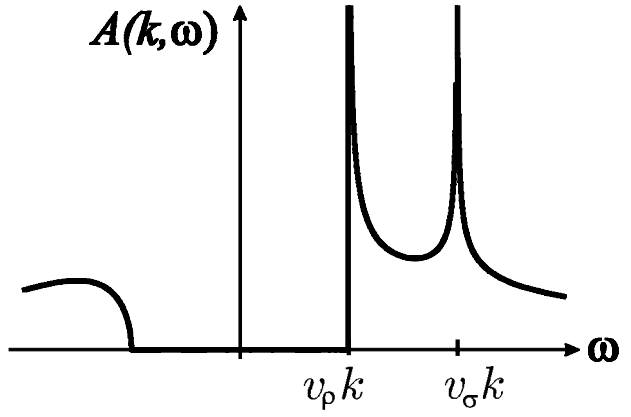


Figure 16. The spectral function of a one dimensional Luttinger liquid. Notice how, in contrast to the Fermi liquid (figure 5), there are now two singular features corresponding to the spinon and holon and they generally disperse with different velocities, v_σ and v_ρ (after Voit (1993) and Dias (1996)).

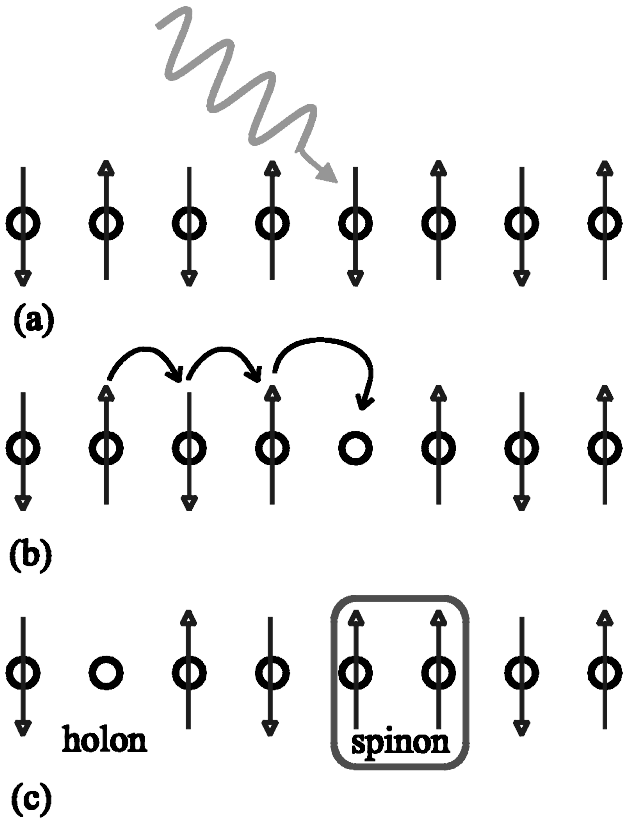


Figure 17. A simple picture of spin–charge separation in one dimension. Consider the 1d *tJ* model when an electron is removed from the antiferromagnetic Mott insulating state by a photon in a photo-emission experiment (*a*). This leaves behind a disruption in both the spin and charge order. (*b*) As electrons move into the vacant site, the locations of the spin and charge disorder separate. They have become distinct particles—a spinon and a holon.

If the essence of the Fermi liquid was the Landau quasiparticle, then the essence of the Luttinger liquid is spin–charge separation and the appearance of spinon and holon quasiparticles. Their existence relies on a very special property of one-dimensional systems: near the Fermi surface all particle–hole excitations at fixed momentum have the same kinetic energy. This is illustrated in figure 15 where we see that in two (and higher dimensions) the energy depends both on the magnitude of q and on its direction relative to the local Fermi surface. In one dimension there is only one direction and so fixing q determines the energy completely. This is important because adding together all the possible particle–hole excitations with a specific excitation momentum $\mathbf{q}$ gives the wavefunction for a density wave: a compression and rarefraction of electron density with a wavelength $2\pi/|\mathbf{q}|$. The special property of one dimension means that here a density wave has a well-defined kinetic energy. Now the potential energy is also usually determined by the density of particles with a given wavelength (see for example the Coulomb interaction of equation (10)) and so the density wave also has a well defined potential energy. This is enough to tell us that density waves form the new eigenstates of the one-dimensional metal. More generally the potential energy can depend on both the density of spin and the density of charge, so spin density waves may have a different energy from charge density waves. Thus the good quantum numbers of the system are those of spin and charge density. So we can completely by-pass the problem of how the electrons behave by working only with the densities.

This leads to the remarkable phenomenon of spin–charge separation. The electron carries with it both spin (its magnetic moment) and its electrical charge. In one dimension these can, and generally do, become two separate entities which move independently as they form the spin and charge density eigenstates. The electron dissolves into its spin part (a spinon) and its charge part (a holon). It is clearly not a Fermi liquid any more because the good quantum numbers look nothing like the old fermion quasiparticle labels. If we ask where the original electron has gone by determining the spectral function, we no longer see the single sharp quasiparticle peak of the Fermi liquid. Instead we see two sharp features characterizing the spin and charge parts of the electron moving with differing velocities (see figure 16).

One can make a very simple picture of how this happens by considering a single electron in a Mott insulating state. This is illustrated in figure 17 where we consider the physics of the tJ model (Box 2) but now in one dimension. Starting with the insulating state we have an antiferromagnetic arrangement of spins. Now we remove an electron which of course removes both a spin and leaves behind a charged state. As the hole now moves we note that the place where the disruption in the spin arrangement and the position of the hole have now moved apart. The spin and charge of the original electron have separated and formed independent entities.

This type of picture has, for many years, seemed no more than a naïve picture of a phenomenon whose proper description requires the powerful mathematical machinery of bosonization. (This is the technique which formally expresses the problem in terms of spin and charge densities.) However, very recently exactly the experiment described above has been done on the antiferromagnetic chain compound $SrCuO_2$ (Kim *et al.* 1996). A single electron is removed from the chain by a photo-emission process whereby an incident photon kicks out an electron. The probability for doing so depends on the underlying dynamics of the spin and charge degrees of freedom we excite. Since this starts as an insulating system the dynamics of the hole are the dynamics of an almost empty band. The allowed momentum of the spinon are restricted by the magnetic order. The spectrum that is seen is shown in figure 18 (*d*). For some momenta of the photo-electron, only the holon is allowed to carry the momentum away and one sees a well defined dispersing peak following the dispersion of the holon. For other momenta, both the spinon and the holon can be excited and so the momentum is distributed between them. Instead of a sharp(ish) quasiparticle peak in the spectrum one sees for these momenta a broader spectrum distributed between the band energies of the spinon and the holon (see figure 18).

The observant reader will perhaps have noticed that the special property upon which all of this relies is a consequence of the linearity of the energy spectrum near a single Fermi point. In reality scattering from near one Fermi surface point to the other side (figure 19 (*a*)) or dispersion curvature as one moves away from the Fermi points (figure 19 (*b*)) means that the momentum of the particle–hole excitation no longer uniquely determines the energy. The Luttinger model is a simplified version of the metallic states that does not contain these troublesome processes (Luttinger 1963). Haldane's Luttinger liquid hypothesis parallels adiabatic continuity in the interacting Fermi liquid but now using the one branch Luttinger model as the starting point. Much as adding interactions to the non-interacting Fermi gas leads to renormalization of (Landau) parameters to form the Landau Fermi-liquid state, in one dimension the additional processes which spoil the special properties of the Luttinger model just lead to a renormalization of the parameters in the model. In fact Haldane showed there were just four free parameters which characterize the low energy properties of one-dimensional metallic states. These properties not only set the values of the spinon and holon velocities but also the so-called anomalous exponents which control, among other properties, the nature of the singularities seen in the electron spectral function.

Identifying unambiguously a Luttinger liquid is made harder by the fact that the thermodynamic probes of specific heat and Pauli susceptibility retain their old Fermi liquid forms even in a Luttinger liquid. Using the resistivity to identify a Luttinger liquid is complicated by the issue of how impurities control the scattering (in one dimension a single impurity limits the current by blocking the current path). In addition one-dimensional systems are typically rather unstable to long range *ordered* spin- and charge-density wave formation which can completely destroy the metallic state. It seems that the most unambiguous probe should be the photo-emission experiment of the type performed above which potentially could measure the spectral function of figure 16 (see Gweon *et al.* 1996, Voit 1998). Nevertheless, from a theoretical perspective this non-Fermi liquid state in 1D is convincingly established and has been found relevant for experiments ranging from the TMTSF Bechgaard salts (Bourbonnais *et al.* 1984, Wzietek *et al.* 1993, Dardel *et al.* 1993, Zwick *et al.* 1997) to quantum wires (Tarucha *et al.* 1995, Yacoby *et al.* 1996) to edge state tunnelling in the fractional quantum Hall effect (Milliken *et al.* 1996, Chang *et al.* 1996).

Phil Anderson has suggested that the Luttinger liquid may not only be confined to the realm of one dimension but

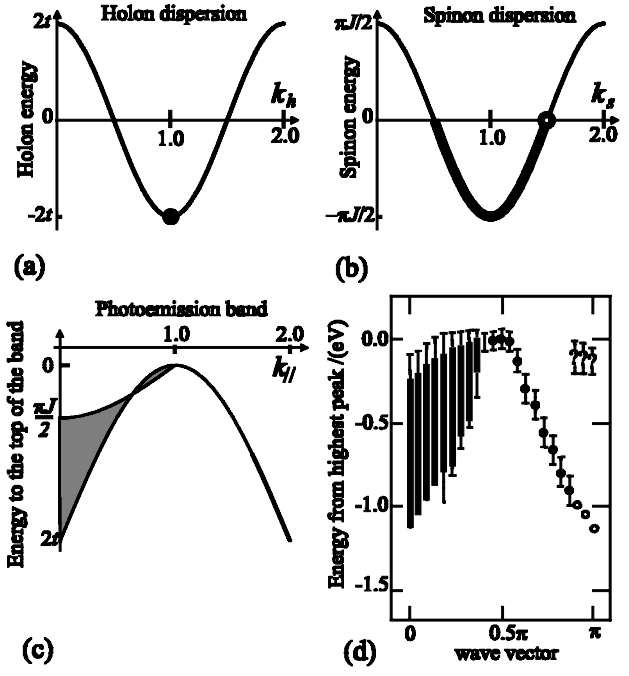


Figure 18. Photo-emission experiments can actually reproduce the physics of the simple picture of figure 17—here in $SrCuO_2$. (*a*) The removal of a single electron creates a holon in an otherwise empty band. (*b*) The magnetic order restricts the allowed momentum of the spinon: the thick line shows the forbidden regions. (*c*) For certain momenta of emitted photo-electron there is just a single way in which the momentum can be distributed between the spinon and holon but in other parts of the zone there is no such restriction. (*d*) In the measurements we see rather broad features where the momentum and energy is distributed between a number of possible spinon and holons states. In other parts of the zone a single dispersing peak is seen (after Kim (1996) and Shen (1997)).

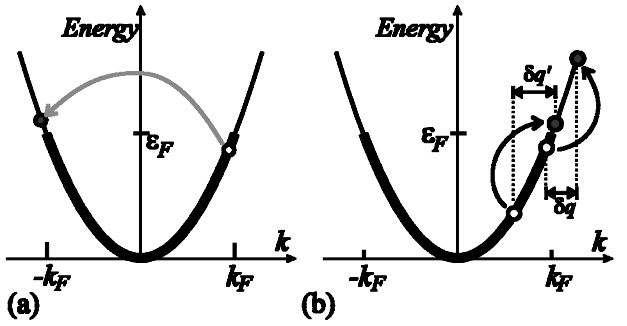


Figure 19. (*a*) Scattering from one Fermi point to the other or (*b*) large energy excitations (and also Umklapp scattering), spoil the special property of 1*d* illustrated in figure 15. The Luttinger liquid hypothesis argues that the density wave eigenstates are still adiabatically continuous with the true low energy eigenstates even in the presence of these processes.

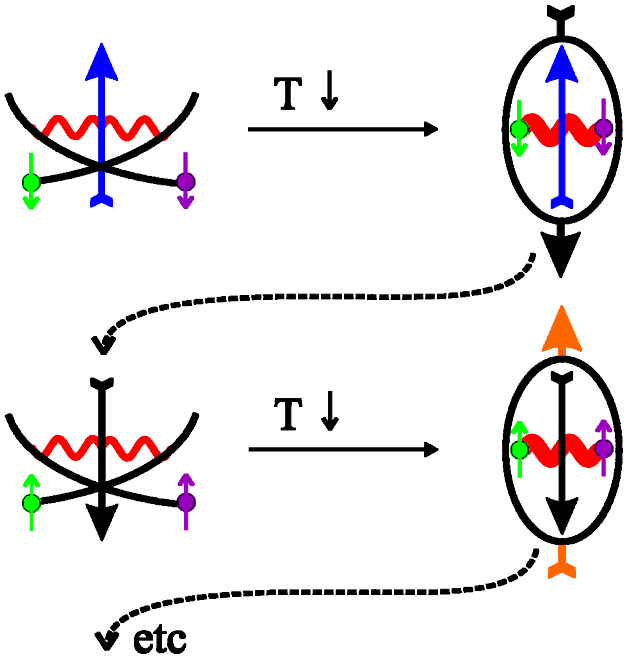


Figure 20. In the two-channel Kondo problem, a single magnetic impurity interacts with *two* orthogonal electron wavefunctions. The magnetic ion can no longer make the usual non-magnetic singlet state at low temperatures because the symmetry of the problem makes favouring one electron over the other impossible. A magnetic state can still undergo further Kondo scattering. The resulting ground state bears no resemblance to a non-interacting gas of electrons and is a local non-Fermi liquid.

may be the appropriate starting point for understanding the two-dimensional metallic state of the cuprates (see Anderson 1997). Part of the supporting experimental evidence which he appeals to is the two separate scattering rates measured in the decay of electrical and Hall currents. These he attributes to the decay of holons and spinons respectively. While this proposal remains controversial, the idea of spin–charge separation in more than one dimension is a very active research area at present.

6. Two-channel Kondo model: novel quasiparticles

Thus far we have had Fermi quasiparticles (in the Fermi liquid), Bose quasiparticles (in the Luttinger liquid) and no quasiparticles at all near a quantum critical point. That might be expected to exhaust the possibilities! In fact in low dimensional systems there can exist excitations which fall outside these classes. I have already mentioned the possibility of particles carrying fractional charge in the fractional quantum Hall effect. The excitations of these systems in two dimensions have quantum statistics that can lie in between fermions and bosons. Under particle exchange these particles acquire a more general phase factor ($\exp(i\theta)$) in contrast to the usual ± 1 for bosons/fermions and are known as 'anyons'—for 'any statistics' (Leinaas and Myrheim 1977)!

Interactions then can lead to completely new types of state appearing at low energies. Rather than use anyons as an example, I will discuss the appearance of a new type of excitation in a metal in the 'two-channel Kondo' problem. The strange particle in this problem is essentially 'half of a spin-half' degree of freedom. Kondo models have for many years been a favourite of condensed matter theorists and the single channel Kondo model is described in Box 1.

If the physics of the single-channel Kondo model is that of a local Fermi liquid, then the two-channel case is the physics of the local *non*-Fermi liquid. In the two-channel case, one imagines a single spin one-half impurity which interacts antiferromagnetically with *two* conduction seas of electrons (hence the two channels) which do not otherwise interact (Nozières and Blandin 1980). The conduction electrons are totally oblivious of the other sea of electrons and do not even experience a Pauli exclusion principle from them. Only the impurity sees that there are two channels. Experimentally this is hard to realize as described above, but there are claims that tunnelling experiments through certain two level systems can be modelled in a very similar way (Ralph *et al*. 1994).

The extra complication of two conduction channels does little to affect the physics at high temperatures. For weak coupling one has a free spin one-half object which scatters both channels of electrons and results in the same logarithmically growing scattering as the temperature is lowered. However, as the coupling constant grows, the impurity spin has a problem. It would like to form a singlet but the symmetry of the problem forbids it from favouring any one of the channels for making that singlet. The other possibility is to make a linear superposition of a singlet with each channel, but this leaves the unbound spin of the spectator channel carrying a two-fold spin degeneracy. It turns out that this then would behave like a new spin-half impurity which in turn wants to undergo another Kondo effect (see figure 20). There is no simple solution to the impurity spin's dilemma. Solving the problem requires the application of conformal field theory techniques (Affleck and Ludwig 1991) and the 'Bethe Ansatz' (Andrei and Destri 1984, Tsvelik and Wiegmann 1985)—a class of wave functions which solve a number of interacting low dimensional problems. The mathematical complexity of these solutions forbids detailed discussion of them here. There do exist a number of 'simplified' treatments (Emery and Kivelson 1992, Sengupta and Georges 1994, Coleman *et al*. 1995) which reformulate the two-channel Kondo model (see Schofield 1997) and make the physical properties obtainable from perturbation theory. What emerges can be seen from the calculated temperature dependence of the impurity spin's entropy. In the single-channel Kondo model this falls smoothly from ln 2 reflecting the two degrees of freedom of the free spin, to 0 at very low temperatures. In the two-channel case the entropy also falls from ln 2. However it saturates at $\frac{1}{2}\ln 2$ at low temperatures as if a $2^{1/2}$ degree of freedom is left. It transpires that this object can be represented rather simply as the real part of the normally complex electron—a so-called Majorana fermion named after Majorana's purely real representation of the Dirac equation (Majorana 1937). It remains free at low temperatures and disrupts the local Fermi liquid one had in the single-channel case.

New power laws emerge for the impurity spin contribution to the low temperature properties: heat capacity $\sim T \ln T$, Pauli susceptibility $\sim \ln T$ and the resistivity has a $T^{1/2}$ correction. While it is hard to make a straightforward application of this model to a physical system, there have been a number of proposals suggesting that this physics can be realized in certain uranium alloys (Cox 1987) as well as in the various tunnelling problems mentioned before. In fact strange power laws are seen in a number of uranium alloys (see, for example, Maple *et al*. 1996), but the two-channel Kondo interpretation remains controversial. Nevertheless it provides us theoretically with the intriguing possibility that the physics of non-Fermi liquids may lead to completely new kinds of low energy particle controlling the behaviour of exotic metals.

7. The disordered Kondo scenario

In our survey of non-Fermi liquids I have deliberately tried to choose examples which are stoichiometric—that is where

a pure crystal exhibits an unusual metallic state. The reason for doing this has been to isolate the role of disorder which further complicates our picture of non-Fermi liquid alloyed materials. Although there exist many powerful techniques for dealing with disorder in condensed matter physics, they unfortunately are not easily extended to include the effect of interactions. Having said that, there are in fact many more examples of non-Fermi liquid metals which one might claim to be disordered. Almost all of the cuprates are made by partially substituting one atom for another in the process of doping the Mott insulating state. (The reason that we believe that these are in fact clean materials is because the copper oxide planes—where the action is taking place—are left unscathed by the doping except for a change in carrier density.) There exist also a growing number of diluted alloys of uranium and cerium which also exhibit non-Fermi liquid behaviour. Some have been interpreted in terms of the two-channel Kondo scenario mentioned above. There is, however, at least one other possibility, namely that the non-Fermi liquid behaviour is coming from *single-channel* Kondo impurities but the disorder creates a distribution of Kondo temperatures (Bernal *et al.* 1995, Miranda *et al.* 1997). One such example is $UCu_{5-x}Pd_x$ (Bernal *et al.* 1995) where one imagines the magnetic uranium ions sitting in a random environment of a Cu/Pd alloy.

I have argued that one such impurity favours a Fermi liquid (see Box 1) so why is it that a distribution of such impurities can lead to anything different? The answer lies in the fact that even relatively weak disorder yields a fraction of the impurity spins with extremely low Kondo temperatures and these magnetic moments remain unquenched and strongly scatter the conduction electrons even at low temperatures. This is a consequence of the exponential dependence of the Kondo temperature on the local properties of the impurity spin:

$$T_{\mathrm{K}} = D\,\exp(-\lambda)\,, \tag{17}$$

where D is the bandwidth of the conduction electrons and λ is a measure of the local density of conduction electronic states at the magnetic site and of the coupling between the moment and the conduction electrons.

We can perform a crude calculation of the non-Fermi liquid properties by assuming that the Kondo temperatures in the alloy are uniformly distributed between 0 and an arbitrary scale, T_0. By using approximate forms for the specific heat, the resistivity and the susceptibility of a single Kondo impurity, we can then simply average over the distribution of Kondo temperatures to obtain the bulk response. Our approximate forms will be

$$C_{\mathrm{imp}} \sim k_{\mathrm{B}}\frac{T_{\mathrm{K}}T}{T^2+T_{\mathrm{K}}^2}\,, \tag{18}$$

$$\chi_{\mathrm{imp}} \sim \frac{\mu_{\mathrm{B}}^2}{T+T_{\mathrm{K}}}\,, \tag{19}$$

$$\rho_{\mathrm{imp}} \sim \begin{cases} 0\,, & T > T_{\mathrm{K}}, \\ \rho_o\,, & T \le T_{\mathrm{K}}. \end{cases} \tag{20}$$

These have been chosen to capture the essence of the results for the Kondo model shown graphically in figure B2 while obeying certain important constraints (such as the total impurity entropy $\int_0^\infty C_{\mathrm{imp}}/T\,\mathrm{d}T$ being independent of T_{k}.) Using a uniform distribution of Kondo temperatures ($P(T_{\mathrm{K}})\mathrm{d}T_{\mathrm{K}} = 1/T_0$) we can straightforwardly calculate the expected properties by averaging over the impurity distribution (e.g. $C(T) \sim N_{\mathrm{imp}}/T_O \int_0^{T_o} C_{\mathrm{imp}}\,dT_{\mathrm{K}}$). One finds that

$$C(T) \sim N_{\mathrm{imp}}k_{\mathrm{B}}\frac{T}{2T_o}\ln\left(1+\frac{T_o^2}{T^2}\right)\,, \tag{21}$$

$$\chi(T) \sim N_{\mathrm{imp}}\frac{\mu_{\mathrm{B}}^2}{T_o}\ln(1+T_o/T)\,, \tag{22}$$

$$\rho(T) \sim \rho_o(1-T/T_o)\,, \quad \text{for } T < T_o\,. \tag{23}$$

We see at once that the specific heat and the susceptibility immediately adopt non-Fermi liquid forms at low temperatures with $C/T \sim \chi \sim -\ln T$ instead of T independent. The rising resistivity as the temperature is lowered suggests disorder in the system as well as a low energy scattering mechanism. Of course these results depend to some extent on the distribution of Kondo temperatures but, provided this distribution tends to a finite number (not zero) at low temperatures, the low temperature forms are robust. In fact this type of behaviour has been seen in many alloyed materials (see Miranda *et al.* (1996) for a table).

To be sure that disorder is driving the non-Fermi liquid physics we would like some independent signature of it. This can come from nuclear magnetic resonance (NMR) and muon spin rotation (μSR) studies (MacLauglin *et al.* 1996). These measurements probe very precisely the local environment at particular atomic sites. What is found is that in $UCu_{5-\chi}Pd_\chi$ the copper atoms appear to sit in a variety of local environments strongly suggesting the presence of disorder. These authors are even able to extract the distribution of Kondo temperatures and show that it does satisfy the requirement of being finite at low temperatures, and is consistent with the measured heat capacity and susceptibilities. While this interpretation is not universally accepted in this particular material (see, for example, Aronson *et al.* 1996) it does serve as the simplest example for a new route to non-Fermi liquid physics when interactions and disorder combine.

8. Conclusions

The discovery of the cuprate superconductors has sparked a widespread interest in materials which do not

seem to lie within the traditional Fermi-liquid framework which we have relied on for understanding the effect of interactions in metals. What is emerging from this is a striking richness in the types of metallic behaviour that can appear. We see metals where the electron dissolves into its magnetic and electric components and systems when no quasiparticle excitation is left. We also see the possibility of unusual states appearing which have no simple analogue outside interacting systems.

In this article I have attempted to give a flavour of some of the ideas which are currently being explored both to understand the cuprates but, often more fruitfully, in understanding equally fascinating problems in other metallic compounds. It may well be that the solution to the mystery of the cuprate metals also lies in some of the physics discussed here. However my suspicion is that nature is playing stranger tricks and that there is a new theory of interacting metals just as profound as Landau's picture of the metallic state that will apply to these compounds.

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